During the first three decades of the twentieth century, our understanding of the physical universe underwent tremendous changes. The classical physics of Newton and the other scientists of the eighteenth and nineteenth centuries was shown to be inadequate to describe completely our universe. The results of this revolution in physics are now called “modern” physics, although they are now almost a century old.

Three of these modern physical concepts are (1) Einstein’s theory of special relativity, which extended Newtonian mechanics; (2) wave-particle duality, which says that both electromagnetic waves and atomic particles have dual wave and particle properties; and (3) quantum mechanics, which revealed that the microscopic atomic world is far different from our everyday macroscopic world. The results and insights provided by these three advances in physics are fundamental to an understanding of nuclear science and technology. This chapter is devoted to describing their basic ideas and results.

2.1 The Special Theory of Relativity

The classical laws of dynamics as developed by Newton were believed, for over 200 years, to describe all motion in nature. Students still spend considerable effort mastering the use of these laws of motion. For example, Newton’s second law, in the form originally stated by Newton, says the rate of change of a body’s momentum $p$ equals the force $F$ applied to it, i.e.,

$$ F = \frac{dp}{dt} = \frac{d(mv)}{dt}. \tag{2.1} $$

For a constant mass $m$, as assumed by Newton, this equation immediately reduces to the modern form of the second law, $F = ma$, where $a = dv/dt$, the acceleration of the body.

In 1905 Einstein discovered an error in classical mechanics and also the necessary correction. In his theory of special relativity,$^1$ Einstein showed that Eq. (2.1) is still correct, but that the mass of a body is not constant, but increases with the body’s speed $v$. The form $F = ma$ is thus incorrect. Specifically, Einstein showed that $m$ increases with $v$ as follows:

$$ m = m_0 \sqrt{1 - \frac{v^2}{c^2}}, $$

where $m_0$ is the mass of the body at rest and $c$ is the speed of light. This equation is known as the relativistic mass equation.

$^1$In 1915 Einstein published the general theory of relativity, in which he generalized his special theory to include gravitation. We will not need this extension in our study of the microscopic world.
varies with the body’s speed as

\[ m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

(2.2)

where \( m_0 \) is the body’s “rest mass,” i.e., the body’s mass when it is at rest, and \( c \) is the speed of light (\( \approx 3 \times 10^8 \) m/s). The validity of Einstein’s correction was immediately confirmed by observing that the electron’s mass did indeed increase as its speed increased in precisely the manner predicted by Eq. (2.2).

Most fundamental changes in physics arise in response to experimental results that reveal an old theory to be inadequate. However, Einstein’s correction to the laws of motion was produced theoretically before being discovered experimentally. This is perhaps not too surprising since in our everyday world the difference between \( m \) and \( m_0 \) is incredibly small. For example, a satellite in a circular earth orbit of 7100 km radius, moves with a speed of 7.5 km/s. As shown in Example 2.1, the mass correction factor \( \sqrt{1 - \frac{v^2}{c^2}} = 1 - 0.31 \times 10^{-9} \), i.e., relativistic effects change the satellite’s mass only in the ninth significant figure or by less than one part a billion! Thus for practical engineering problems in our macroscopic world, relativistic effects can safely be ignored. However, at the atomic and nuclear level, these effects can be very important.

Example 2.1: What is the fractional increase in mass of a satellite traveling at a speed of 7.5 km/s? From Eq. (2.2) find the fractional mass increase to be

\[ \frac{m - m_0}{m_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1. \]

Here \( \frac{v^2}{c^2} = (7.5 \times 10^3 / 2.998 \times 10^8)^2 = 6.258 \times 10^{-10} \). With this value of \( \frac{v^2}{c^2} \) most calculators will return a value of 0 for the fractional mass increase. Here’s a trick for evaluating relativistically expressions for such small values of \( \frac{v^2}{c^2} \).

The expression \((1 + \epsilon)^n\) can be expanded in a Taylor series as

\[ (1 + \epsilon)^n = 1 + n\epsilon + \frac{n(n + 1)}{2!}\epsilon^2 + \frac{n(n - 1)(n - 2)}{3!}\epsilon^3 + \cdots \approx 1 + n\epsilon \quad \text{for} \ |\epsilon| << 1. \]

Thus, with \( \epsilon = -\frac{v^2}{c^2} \) and \( n = -1/2 \) we find

\[ \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2} \frac{v^2}{c^2} \]

so that the fractional mass increase is

\[ \frac{m - m_0}{m_0} \approx \left[\frac{1}{2} \frac{v^2}{c^2}\right] = 3.12 \times 10^{-9}. \]
Example 2.1

Problem: What is the fractional increase in the mass of a satellite traveling at a speed of 7.5 km/s?

Solution: Using Eq. (2.2) of the text we have:

\[ \text{fiim} = \frac{m - m_0}{m_0} \]

dimensionless, Fractional increase in mass

As always when working this type of problem, the speed of light must be considered in addition to the observed speed of the specific object in question. Therefore we have:

\[ c := 2.998 \cdot 10^8 \quad \text{m/s, Speed of light} \]
\[ v := 7.5 \cdot 10^3 \quad \text{m/s, Speed of satellite} \]

\[ \text{fiim} := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \quad \text{dimensionless, Fractional increase in mass} \]

\[ \text{fiim} = 3.1292 \cdot 10^{-10} \]

This concludes the problem solution for Example Problem 2.1!
2.1.1 Principle of Relativity

The principle of relativity is older than Newton’s laws of motion. In Newton’s words (actually translated from Latin) “The motions of bodies included in a given space are the same amongst themselves, whether the space is at rest or moves uniformly forward in a straight line.” This means that experiments made in a laboratory in uniform motion (e.g., in a non-accelerating train) produce the same results as when the laboratory is at rest. Indeed this principle of relativity is widely used to solve problems in mechanics by shifting to moving frames of reference to simplify the equations of motion.

The relativity principle is a simple intuitive and appealing idea. But do all the laws of physics indeed remain the same in all non-accelerating (inertial) coordinate systems? Consider the two coordinate systems shown in Fig. 2.1. System S is at rest, while system S’ is moving uniformly to the right with speed v. At t = 0, the origin of S’ is at the origin of S. The coordinates of some point P are (x, y, z) in S and (x’, y’, z’) in S’. Clearly, the primed and unprimed coordinates are related by

\[ x' = x - vt; \quad y' = y; \quad z' = z; \quad \text{and} \quad t' = t. \]  

(2.3)

If these coordinate transformations are substituted into Newton’s laws of motion, we find they remain the same. For example, consider a force in the x-direction, \( F_x \), acting on some mass \( m \). Then the second law in the S’ moving system is \( F_x = m \frac{d^2x'}{dt'^2} \). Now transform this law to the stationary S system. We find (for \( v \) constant)

\[ F_x = m \frac{d^2x'}{dt'^2} = m \frac{d^2(x - vt)}{d(t)^2} = m \frac{d^2x}{dt^2}. \]

Thus the second law has the same form in both systems. Since the laws of motion are the same in all inertial coordinate systems, it follows that it is impossible to tell, from results of mechanical experiments, whether or not the system is moving.

In the 1870s, Maxwell introduced his famous laws of electromagnetism. These laws explained all observed behavior of electricity, magnetism, and light in a uniform system. However, when Eqs. (2.3) are used to transform Maxwell’s equations to another inertial system, they assume a different form. Thus from optical experiments in a moving system, one should be able to determine the speed of the system. For many years Maxwell’s equations were thought to be somehow incorrect, but 20 years of research only continued to reconfirm them. Eventually, some scientists began to wonder if the problem lay in the Galilean transformation of Eqs. (2.3). Indeed, Lorentz observed in 1904 that if the transformation

\[ x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}; \quad y' = y; \quad z' = z; \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \]

(2.4)

is used, Maxwell’s equations become the same in all inertial coordinate systems. Poincaré, about this time, even conjectured that all laws of physics should re-
main unchanged under the peculiar looking Lorentz transformation. The Lorentz transformation is indeed strange, since it indicates that space and time are not independent quantities. Time in the $S'$ system, as measured by an observer in the $S$ system, is different from the time in the observer’s system.

2.1.2 Results of the Special Theory of Relativity

It was Einstein who, in 1905, showed that the Lorentz transformation was indeed the correct transformation relating all inertial coordinate systems. He also showed how Newton’s laws of motion must be modified to make them invariant under this transformation.

Einstein based his analysis on two postulates:

- The laws of physics are expressed by equations that have the same form in all coordinate systems moving at constant velocities relative to each other.

- The speed of light in free space is the same for all observers and is independent of the relative velocity between the source and the observer.

The first postulate is simply the principle of relativity, while the second states that we observe light to move with speed $c$ even if the light source is moving with respect to us. From these postulates, Einstein demonstrated several amazing properties of our universe.

1. The laws of motion are correct, as stated by Newton, if the mass of an object is made a function of the object’s speed $v$, i.e.,

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}.$$  \hspace{1cm} (2.5)

This result also shows that no material object can travel faster than the speed of light since the relativistic mass $m(v)$ must always be real. Further, an object with a rest mass ($m_0 > 0$) cannot even travel at the speed of light; otherwise its relativistic mass would become infinite and give it an infinite kinetic energy.

2. The length of a moving object in the direction of its motion appears smaller to an observer at rest, namely

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}.$$ \hspace{1cm} (2.6)

where $L_0$ is the “proper length” or length of the object when at rest.

3. The passage of time appears to slow in a system moving with respect to a stationary observer. The time $t$ required for some physical phenomenon (e.g., the interval between two heart beats) in a moving inertial system appears to be longer (dilated) than the time $t_0$ for the same phenomenon to occur in the stationary system. The relation between $t$ and $t_0$ is

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}.$$ \hspace{1cm} (2.7)
4. Perhaps the most famous result from special relativity is the demonstration of the equivalence of mass and energy by the well-known equation

\[ E = mc^2. \]  

(2.8)

This result says energy and mass can be converted to each other. Indeed, all changes in energy of a system result in a corresponding change in the mass of the system. This equivalence of mass and energy plays a critical role in the understanding of nuclear technology.

The first three of these results are derived in the Addendum 1 to this chapter. The last result, however, is so important that it is derived below.

**Derivation of** \( E = mc^2 \)

Consider a particle with rest mass \( m_0 \) initially at rest. At time \( t = 0 \) a force \( F \) begins to act on the particle accelerating the mass until at time \( t \) it has acquired a velocity \( v \) (see Fig. 2.2). From the conservation-of-energy principle, the work done on this particle as it moves along the path of length \( s \) must equal the kinetic energy \( T \) of the particle at the end of the path. The path along which the particle moves is arbitrary, depending on how \( F \) varies in time. The work done by \( F \) (a vector) on the particle as it moves through a displacement \( ds \) (also a vector) is \( F \cdot ds \). The total work done on the particle over the whole path of length \( s \) is

\[ T = \int_0^s F \cdot ds = \int_0^s \frac{d(mv)}{dt} \cdot ds = \int_0^t \frac{d(mv)}{dt} \cdot \frac{ds}{dt} dt. \]

But, \( ds/dt = v \), a vector parallel to \( d(mv)/dt \); thus,

\[ T = \int_0^t \frac{d(mv)}{dt} v dt = \int_0^m v d(mv). \]

Substitution of Eq. (2.5) for \( m \) yields

\[
\begin{align*}
T &= m_0 \int_0^v v d \left( \frac{v}{\sqrt{1 - v^2/c^2}} \right) \\
&= m_0 \int_0^v v \left( \frac{1}{\sqrt{1 - v^2/c^2}} + \frac{v^2/c^2}{(1 - v^2/c^2)^{3/2}} \right) dv \\
&= m_0 \int_0^v \frac{v}{(1 - v^2/c^2)^{3/2}} dv \\
&= m_0 c^2 \left. \frac{1}{\sqrt{1 - v^2/c^2}} \right|_0^v \\
&= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2, \quad (2.9)
\end{align*}
\]
or finally

\[ T = mc^2 - m_0c^2. \] (2.10)

Thus we see that the kinetic energy is associated with the increase in the mass of the particle.

Equivalently, we can write this result as \( m_0c^2 = m_0c^2 + T \). We can interpret \( m_0c^2 \) as the particle’s "total energy" \( E \), which equals its rest-mass energy plus its kinetic energy. If the particle was also in some potential field, for example, an electric field, the total energy would also include the potential energy. Thus we have

\[ E = mc^2. \] (2.11)

This well known equation is the cornerstone of nuclear energy analyses. It shows the equivalence of energy and mass. One can be converted into the other in precisely the amount specified by \( E = mc^2 \). When we later study various nuclear reactions, we will see many examples of energy being converted into mass and mass being converted into energy.

**Example 2.2:** What is the energy equivalent in MeV of the electron rest mass?

From data in Table 1.5 and Eq. 1.1 we find

\[
E = m_e c^2 = (9.109 \times 10^{-31} \text{ kg}) \times (2.998 \times 10^8 \text{ m/s})^2 \\
\times (1 \text{ J/(kg m}^2 \text{ s}^{-2})/(1.602 \times 10^{-13} \text{ J/MeV}) \\
= 0.5110 \text{ MeV}
\]

When dealing with masses on the atomic scale, it is often easier to use masses measured in atomic mass units (u) and the conversion factor of 931.49 MeV/u. With this important conversion factor we obtain

\[
E = m_e c^2 = (5.486 \times 10^{-4} \text{ u}) \times (931.49 \text{ MeV/u}) = 0.5110 \text{ MeV}.
\]

**Reduction to Classical Mechanics**

For slowly moving particles, that is, \( v \ll c \), Eq. (2.10) yields the usual classical result. Since,

\[
\frac{1}{\sqrt{1 - v^2/c^2}} \equiv (1 - v^2/c^2)^{-1/2} = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \cdots \simeq 1 + \frac{v^2}{2c^2},
\] (2.12)

the kinetic energy of a slowly moving particle is

\[
T = m_0c^2 \left( \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) = m_0c^2 \left( \left[ 1 + \frac{v^2}{2c^2} + \cdots \right] - 1 \right) \simeq \frac{1}{2} m_0v^2. \] (2.13)

Thus the relativistic kinetic energy reduces to the classical expression for kinetic energy if \( v \ll c \), a reassuring result since the validity of classical mechanics is well established in the macroscopic world.
Example 2.2

Problem: What is the energy equivalent in MeV of the electron rest mass?

Solution: We look up the electron rest mass in Appendix A.1 and using Eq. (2.11) of the text we have:

\[
\begin{align*}
c &:= 2.998 \cdot 10^8 \text{ m/s, Speed of light} \\
me &:= 9.1093819 \cdot 10^{-31} \text{ kg, Electron rest mass} \\
E_{\text{MeV}} &:= \frac{me \cdot c^2}{1.60217646 \cdot 10^{-13}} \text{ MeV, Energy equivalent when using conversion factor of } 1.602 \cdot 10^{-19} \text{ J/eV}
\end{align*}
\]

\[E_{\text{MeV}} = 0.5110\]

This concludes the problem solution for Example Problem 2.2!
Relation Between Kinetic Energy and Momentum

Both classically and relativistically the momentum \( p \) of a particle is given by,

\[
p = mv.
\]  
(2.14)

In classical physics, a particle’s kinetic energy \( T \) is given by,

\[
T = \frac{mv^2}{2} = \frac{p^2}{2m},
\]

which yields

\[
p = \sqrt{2mT}.
\]  
(2.15)

For relativistic particles, the relationship between momentum and kinetic energy is not as simple. Square Eq. (2.5) to obtain

\[
p^2 = (mv)^2 = (mc)^2 = \frac{1}{c^2}[(mc^2)^2 - (m_0c^2)^2].
\]

Then combine this result with Eq. (2.10) to obtain

\[
p^2 = \frac{1}{c^2} [(T + m_0c^2)^2 - (m_0c^2)^2] = \frac{1}{c^2} [T^2 + 2Tm_0c^2].
\]  
(2.16)

Thus for relativistic particles

\[
p = \frac{1}{c} \sqrt{T^2 + 2Tm_0c^2}.
\]  
(2.17)

Particles

For most moving objects encountered in engineering analyses, the classical expression for kinetic energy can be used. Only if an object has a speed near \( c \) must we use relativistic expressions. From Eq. (2.10) one can readily calculate the kinetic energies required for a particle to have a given relativistic mass change. Listed in Table 2.1 for several important atomic particles are the rest mass energies and the kinetic energies required for a 0.1% mass change. At this threshold for relativistic effects, the particle’s speed \( v = 0.045c \) (see Problem 2).

2.2 Radiation as Waves and Particles

For many phenomena, radiant energy can be considered as electromagnetic waves. Indeed Maxwell’s equations, which describe very accurately interactions of long wave-length radiation, readily yield a wave equation for the electric and magnetic fields of radiant energy. Phenomena such as diffraction, interference, and other related optical effects can be described only by a wave model for radiation.
Table 2.1. Rest mass energies and kinetic energies for a 0.1% relativistic mass increase for four particles.

<table>
<thead>
<tr>
<th>Particle</th>
<th>rest mass energy $m_0c^2$</th>
<th>kinetic energy for a 0.1% increase in mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>electron</td>
<td>0.511 MeV</td>
<td>511 eV $\approx 0.5$ keV</td>
</tr>
<tr>
<td>proton</td>
<td>938 MeV</td>
<td>938 keV $\approx 1$ MeV</td>
</tr>
<tr>
<td>neutron</td>
<td>940 MeV</td>
<td>940 keV $\approx 1$ MeV</td>
</tr>
<tr>
<td>$\alpha$-particle</td>
<td>3751 MeV</td>
<td>3.8 MeV $\approx 4$ MeV</td>
</tr>
</tbody>
</table>

However, near the beginning of the twentieth century, several experiments involving light and X rays were performed that indicated that radiation also possessed particle-like properties. Today we understand through quantum theory that matter (e.g., electrons) and radiation (e.g., X rays) both have wave-like and particle properties. This dichotomy, known as the \textit{wave-particle duality principle}, is a cornerstone of modern physics. For some phenomena, a wave description works best; for others, a particle model is appropriate. In this section, three pioneering experiments are reviewed that helped to establish the wave-particle nature of matter.

2.2.1 The Photoelectric Effect

In 1887, Hertz discovered that, when metal surfaces were irradiated with light, "electricity" was emitted. J.J. Thomson in 1898 showed that these emissions were electrons (thus the term \textit{photoelectrons}). According to a classical (wave theory) description of light, the light energy was absorbed by the metal surface, and when sufficient energy was absorbed to free a bound electron, a photoelectron would "boil" off the surface. If light were truly a wave, we would expect the following observations:

- Photoelectrons should be produced by light of all frequencies.
- At low intensities a time lag would be expected between the start of irradiation and the emission of a photoelectron since it takes time for the surface to absorb sufficient energy to eject an electron.
- As the light intensity (i.e., wave amplitude) increases, more energy is absorbed per unit time and, hence, the photoelectron emission rate should increase.
- The kinetic energy of the photoelectron should increase with the light intensity since more energy is absorbed by the surface.

However, experimental results differed dramatically with these results. It was observed:

- For each metal there is a minimum light frequency below which no photoelectrons are emitted no matter how high the intensity.
- There is no time lag between the start of irradiation and the emission of photoelectrons, no matter how low the intensity.
The intensity of the light affects only the emission rate of photoelectrons.

The kinetic energy of the photoelectron depends only on the frequency of the light and not on its intensity. The higher the frequency, the more energetic is the photoelectron.

In 1905 Einstein introduced a new light model which explained all these observations. Einstein assumed that light energy consists of photons or “quanta of energy,” each with an energy \( E = h \nu \), where \( h \) is Planck’s constant \((6.62 \times 10^{-34} \text{ J s})\) and \( \nu \) is the light frequency. He further assumed that the energy associated with each photon interacts as a whole, i.e., either all the energy is absorbed by an atom or none is. With this “particle” model, the maximum kinetic energy of a photoelectron would be

\[
E = h \nu - A, \tag{2.18}
\]

where \( A \) is the amount of energy (the so-called work function) required to free an electron from the metal. Thus if \( h \nu < A \), no photoelectrons are produced. Increasing the light intensity only increases the number of photons hitting the metal surface per unit time and, thus, the rate of photoelectron emission.

Although Einstein was able to explain qualitatively the observed characteristics of the photoelectric effect, it was several years later before Einstein’s prediction of the maximum energy of a photoelectron, Eq. (2.18), was verified quantitatively using the experiment shown schematically in Fig. 2.3. Photoelectrons emitted from freshly polished metallic surfaces were absorbed by a collector causing a current to flow between the collector and the irradiated metallic surface. As an increasing negative voltage was applied to the collector, fewer photoelectrons had sufficient kinetic energy to overcome this potential difference and the photoelectric current decreased to zero at a critical voltage \( V_0 \) at which no photoelectrons had sufficient kinetic energy to overcome the opposing potential. At this voltage, the maximum kinetic energy of a photoelectron, Eq. (2.18), equals the potential energy \( V_0 e \) the photoelectron must overcome, i.e.,

\[
V_0 e = h \nu - A,
\]

or

\[
V_0 = \frac{h \nu}{e} - \frac{A}{e}, \tag{2.19}
\]

where \( e \) is the electron charge. In 1912 Hughes showed that, for a given metallic surface, \( V_0 \) was a linear function of the light frequency \( \nu \). In 1916 Milliken, who had previously measured the electron charge \( e \), verified that plots of \( V_0 \) versus \( \nu \) for different metallic surfaces had a slope of \( h/e \), from which \( h \) could be evaluated.

\[2\text{It is an interesting historical fact that Einstein received the Nobel prize for his photoelectric research and not for his theory of relativity, which he produced in the same year.}\]
Milliken’s value of $h$ was in excellent agreement with the value determined from measurements of black-body radiation, in whose theoretical description Planck first introduced the constant $h$.

The prediction by Einstein and its subsequent experimental verification clearly demonstrated the quantum nature of radiant energy. Although the wave theory of light clearly explained diffraction and interference phenomena, scientists were forced to accept that the energy of electromagnetic radiation could somehow come together into individual quanta, which could enter an individual atom and be transferred to a single electron. This quantization occurs no matter how weak the radiant energy.

**Example 2.3:** What is the maximum wavelength of light required to liberate photoelectrons from a metallic surface with a work function of 2.35 eV (the energy able to free a valence electron)? At the minimum frequency, a photon has just enough energy to free an electron. From Eq. (2.18) the minimum frequency to yield a photon with zero kinetic energy ($E=0$) is

$$\nu_{\text{min}} = A/h = 2.35 \text{ eV}/4.136 \times 10^{-15} \text{ eV}/\text{s} = 5.68 \times 10^{14} \text{ s}^{-1}. $$

The wavelength of such radiation is

$$\lambda_{\text{max}} = c/\nu_{\text{min}} = 2.998 \times 10^8 \text{ m s}^{-1}/5.68 \times 10^{14} \text{ s}^{-1} = 5.28 \times 10^{-7} \text{ m}. $$

This corresponds to light with a wavelength of 528 nm which is in the green portion of the visible electromagnetic spectrum.

### 2.2.2 Compton Scattering

Other experimental observations showed that light, besides having quantized energy characteristics, must have another particle-like property, namely momentum. According to the wave model of electromagnetic radiation, radiation should be scattered from an electron with no change in wavelength. However, in 1922 Compton observed that x rays scattered from electrons had a decrease in the wavelength $\Delta \lambda = \lambda' - \lambda$ proportional to $(1 - \cos \theta_s)$ where $\theta_s$ was the scattering angle (see Fig. 2.4). To explain this observation, it was necessary to treat x rays as particles with a linear momentum $p = h/\lambda$ and energy $E = h\nu = pc$.

In an x-ray scattering interaction, the energy and momentum before scattering must equal the energy and momentum after scattering. Conservation of linear momentum requires the initial momentum of the incident photon (the electron is assumed to be initially at rest) to equal the vector sum of the momenta of the scattered photon and the recoil electron. This requires the momentum vector triangle of Fig. 2.5 to be closed, i.e.,

$$\mathbf{p}_\lambda = \mathbf{p}_\lambda + \mathbf{p}_e \quad (2.20)$$

or from the law of cosines

$$p_e^2 = p_\lambda^2 + p_\lambda^2 - 2p_\lambda p_\lambda \cos \theta_s. \quad (2.21)$$
Example 2.3

Problem: What is the maximum wavelength of light required to liberate photoelectrons from a metallic surface with a work function of 2.35 eV (the energy necessary to free a valence electron)?

Solution: At the minimum frequency, a photon has just enough energy to free an electron. Using Eq. (2.18) the minimum frequency to yield a photon with zero kinetic energy \( E = 0 \) is:

\[
\nu_{\text{min}} = \frac{A}{h}
\]

\[
\nu_{\text{min}} = 5.6818 \times 10^{14} \text{ s}^{-1}, \text{ Minimum frequency}
\]

Now we can find the wavelength by using Eq. (2.28) as follows:

\[
\lambda := \frac{c}{\nu_{\text{min}}}
\]

\[
\lambda = 5.2765 \times 10^{-7} \text{ m}, \text{ Wavelength of light}
\]

\[
\lambda = 5.2765 \times 10^{-7} \text{ m}, \text{ Note this is green light in the visible electromagnetic spectrum!}
\]

This concludes the problem solution for Example Problem 2.3!
A photon with wavelength $\lambda$ is scattered by an electron. After scattering, the photon has a longer wavelength $\lambda'$ and the electron recoils with an energy $T_e$ and momentum $p_e$.

The conservation of energy requires

$$p_\lambda c + m_e c^2 = p'_\lambda c + m e'^2$$

(2.22)

where $m_e$ is the rest-mass of the electron before the collision when it has negligible kinetic energy, and $m$ is its relativistic mass after scattering the photon. This result, combined with Eq. (2.16) (in which $m_e = m_o$), can be rewritten as

$$p_\lambda + m_e c - p_\lambda = \sqrt{p_e^2 + (m_e c)^2}.$$  

(2.23)

Substitute for $p_e$ from Eq. (2.21) into Eq. (2.23), square the result, and simplify to obtain

$$\frac{1}{p'_\lambda} - \frac{1}{p_\lambda} = \frac{1}{m_e c} (1 - \cos \theta_e).$$

(2.24)

Then since $\lambda = h/p$, this result gives the decrease in the scattered wavelength as

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta_e),$$

(2.25)

where $h/(m_e c) = 2.431 \times 10^{-6} \mu m$. Thus, Compton was able to predict the wavelength change of scattered x rays by using a particle model for the x rays, a prediction which could not be obtained with a wave model.

This result can be expressed in terms of the incident and scattered photon energies, $E$ and $E'$, respectively. With the photon relations $\lambda = c/\nu$ and $E = h\nu$, Eq. (2.25) gives

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2} (1 - \cos \theta_e).$$

(2.26)
Example 2.4: What is the recoil kinetic energy of the electron that scatters a 3-MeV photon by 45 degrees? In such a Compton scattering event, we first calculate the energy of the scattered photon. From Eq. (2.26) the energy $E'$ of the scattered photon is found to be

$$E' = \frac{1}{E + \frac{1}{m_e c^2}} \left[ 1 - \cos \theta_s \right]^{-1} = \sqrt{\frac{1}{3 \text{ MeV}} + \frac{1}{0.511 \text{ MeV}} \left[ 1 - \cos \left( \frac{\pi}{4} \right) \right]} = 1.10 \text{ MeV}. $$

Because energy is conserved, the kinetic energy $T_e$ of the recoil electron must equal the energy lost by the photon, i.e., $T_e = E - E' = 3 - 1.10 = 1.90 \text{ MeV}$.

2.2.3 Electromagnetic Radiation: Wave-Particle Duality

Electromagnetic radiation assumes many forms encompassing radio waves, microwaves, visible light, X rays, and gamma rays. Many properties are described by a wave model in which the wave travels at the speed of light $c$ and has a wavelength $\lambda$ and frequency $\nu$, which are related by the wave speed formula

$$c = \lambda \nu. \tag{2.27} $$

The wave properties account for many phenomena involving light such as diffraction and interference effects.

However, as Einstein and Compton showed, electromagnetic radiation also has particle-like properties, namely, the light energy being carried by discrete quanta or packets of energy called photons. Each photon has an energy $E = h\nu$ and interacts with matter (atoms) in particle-like interactions (e.g., in the photoelectric interactions described above).

Thus, light has both wave-like and particle-like properties. The properties or model we use depend on the wavelength of the radiation being considered. For example, if the wavelength of the electromagnetic radiation is much longer than the dimensions of atoms $\sim 10^{-10}$ m (e.g., visible light, infrared radiation, radar and radio waves), the wave model is usually most useful. However, for short wavelength electromagnetic radiation $\lesssim 10^{-12}$ m (e.g., ultraviolet, x rays, gamma rays), the corpuscular or photon model is usually used. This is the model we will use in our study of nuclear science and technology, which deals primarily with penetrating short-wavelength electromagnetic radiation.

Photon Properties

Some particles must always be treated relativistically. For example, photons, by definition, travel with the speed of light $c$. From Eq. (2.5), one might think that photons have an infinite relativistic mass, and hence, from Eq. (2.17), infinite momentum. This is obviously not true since objects, when irradiated with light, are not observed to jump violently. This apparent paradox can easily be resolved if we insist that the rest mass of the photon be exactly zero, although its relativistic mass
Problem: What is the recoil kinetic energy of the electron that scatters a 3-MeV photon by 45 degrees?

Solution: In such a Compton scattering event, we first calculate the energy of the scattered photon. Using Eq. (2.26) the energy $E'$ of the scattered photon is calculated to be:

$$E := 3 \text{ MeV, Energy of the scattered photon}$$

$$mec2 := 0.5110 \text{ MeV, Rest energy of an electron}$$

$$\theta_s := 45 \cdot \frac{\pi}{180} \text{ radians, Angle of scatter}$$

$$E' := \left[ \frac{1}{E} + \frac{1}{mec2} \left(1 - \cos(\theta_s)\right) \right]^{-1} \text{ MeV, Energy of the incident (i.e., striker) photon}$$

$$E' = 1.1031$$

Because energy is conserved, the kinetic energy $T_e$ of the recoil electron must be a result of the energy lost by the photon, as calculated below:

$$T_e := E - E' \text{ MeV, Kinetic energy of the recoil electron}$$

$$T_e = 1.8969$$

This concludes the problem solution for Example Problem 2.4!
is finite. In fact, the total energy of a photon, \( E = h\nu \), is due strictly to its motion. Equation (2.17) immediately gives the momentum of a photon (with \( m_0 \equiv 0 \)) as,

\[
p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda},
\]

From Eq. (2.10), the photon's relativistic mass is,

\[
m = \sqrt{E^2 - c^2} = \frac{h\nu}{c^2},
\]

or

\[
m = \frac{h\nu}{c^2}.
\]

### 2.2.4 Electron Scattering

In 1924 de Broglie postulated that, since light had particle properties, then for symmetry (physicists love symmetry!), particles should have wave properties. Because photons had a discrete energy \( E = h\nu \) and momentum \( p = h/\lambda \), de Broglie suggested that a particle, because of its momentum, should have an associated wavelength \( \lambda = h/p \).

![Figure 2.6](image)

**Figure 2.6.** Electrons scattering from atoms on a crystalline plane, interfere constructively if the distance \( AB \) is a multiple of the electron's de Broglie wavelength.

![Figure 2.7](image)

**Figure 2.7.** Observed number of electrons \( N(\theta) \) scattered into a fixed cone or directions about an angle \( \theta \) by the atoms in a nickel crystal.

Davisson and Germer in 1927 confirmed that electrons did indeed behave like waves with de Broglie's predicted wavelength. In their experiment, shown schematically in Fig. 2.6, Davisson and Germer illuminated the surface of a Ni crystal by a perpendicular beam of 54-eV electrons and measured the number of electrons \( N(\theta) \) reflected at different angles \( \theta \) from the incident beam. According to the particle model, electrons should be scattered by individual atoms isotropically and \( N(\theta) \) should exhibit no structure. However, \( N(\theta) \) was observed to have a peak near 50° (see Fig. 2.7). This observation could only be explained by recognizing the peak as a constructive interference peak — a wave phenomenon. Specifically, two reflected electron waves are in phase (constructively interfere) if the difference in their path lengths \( AB \) in Fig. 2.6 is an integral number of wavelengths, i.e., if \( d\sin\theta = n\lambda \), \( n = 1, 2, \ldots \) where \( d \) is the distance between atoms of the crystal. This experiment and many similar ones clearly demonstrated that electrons (and other particles such as atoms) have wave-like properties.
2.2.5 Wave-Particle Duality

The fact that particles can behave like waves and that electromagnetic waves can behave like particles seems like a paradox. What really is a photon or an electron? Are they waves or particles? The answer is that entities in nature are more complex than we are used to thinking, and they have, simultaneously, both particle and wave properties. Which properties dominate, depends on the object’s energy and mass. In Fig. 2.8, the de Broglie wavelength

\[ \lambda = \frac{h}{p} = \frac{hc}{\sqrt{T^2 + 2Tm_0c^2}} \]  

(2.30)

is shown for several objects as the kinetic energy \( T \) increases. For a classical object \( (T^2 << 2Tm_0c^2) \), the wavelength is given by \( \lambda = \frac{h}{\sqrt{2m_0T}} \). However, as the object’s speed increases, its behavior eventually becomes relativistic \( (T >> 2m_0c^2) \) and the wavelength varies as \( \lambda = \frac{h}{T} \), the same as that for a photon. When the wavelength of an object is much less than atomic dimensions \( (\sim 10^{-10} \text{ m}) \), it behaves more like a classical particle than a wave. However, for objects with longer wavelengths, wave properties tend to be more apparent than particle properties.

Figure 2.8. The de Broglie wavelength for several objects as their kinetic energy varies. Shown are the electron \((e^-)\), a hydrogen atom \((H)\), an atom of \(^{238}\text{U}\) \((U)\), a 1-mg fruit fly, and a 2000-lb car. For wavelengths large than atomic sizes, the objects’ wave-like behavior dominates, while for smaller wavelengths, the objects behave like particles. On the right, regions of the electromagnetic spectrum are indicated.
Example 2.5] What is the de Broglie wavelength of a neutron with kinetic energy $T$ in eV? We saw earlier that a neutron with less than about 10 MeV of kinetic energy can be treated classically, i.e., its momentum is given by $p = \sqrt{2m_nT}$. Thus, the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_nT}}.$$  

Since $T$ is in eV, we express $m_n$ and $h$ in eV units. From Table 1.3 we find $h = 4.136 \times 10^{-15} \text{ eV s}$ and $m_n = 939.6 \times 10^6 \text{ eV/c}^2 = 939.6 \times 10^6 \text{ eV}/(2.998 \times 10^8 \text{ m/s})^2 = 1.045 \times 10^{-8} \text{ eV m}^{-2} \text{ s}^2$. Thus

$$\lambda = \frac{4.136 \times 10^{-15} \text{ eV s}}{\sqrt{2 \times 1.045 \times 10^{-8} T \text{ eV}^2 \text{ m}^{-2} \text{ s}^2}} = \frac{2.860 \times 10^{-11} \text{ m}}{\sqrt{T}}.$$  

For very low energy neutrons, e.g., $10^{-6} \text{ eV}$ with a wavelength of $2.86 \times 10^{-8} \text{ m}$, the “size” of the neutron is comparable to the distances between atoms in a molecule, and such neutrons can scatter from several adjacent atoms simultaneously and create a neutron diffraction pattern from which the geometric structure of molecules and crystals can be determined. By contrast, neutrons with energies of 1 MeV have a wavelength of $2.86 \times 10^{-14} \text{ m}$, comparable to the size of a nucleus. Thus, such neutrons can interact only with a single nucleus. At even higher energies, the wavelength becomes even smaller and a neutron begins to interact with individual nucleons inside a nucleus.

2.3 Quantum Mechanics

The demonstration that particles (point objects) also had wave properties led to another major advance of modern physics. Because a material object such as an electron has wave properties, it should obey some sort of wave equation. Indeed, Schrödinger in 1925 showed that atomic electrons could be well described as standing waves around the nucleus. Further, the electron associated with each wave could have only a discrete energy. The branch of physics devoted to this wave description of particles is called quantum mechanics or wave mechanics.

2.3.1 Schrödinger’s Wave Equation

To illustrate Schrödinger’s wave equation, we begin with an analogy to the standing waves produced by a plucked string anchored at both ends. The wave equation that describes the displacement $\Psi(x, t)$ as a function of position $x$ from one end of the string, which has length $L$, and at time $t$ is

$$\frac{\partial^2 \Psi(x, t)}{\partial x^2} = \frac{1}{u^2} \frac{\partial^2 \Psi(x, t)}{\partial t^2}. \tag{2.31}$$

Here $u$ is the wave speed. There are infinitely many discrete solutions to this homogeneous partial differential equation, subject to the boundary condition $\Psi(0, t) = \Psi(L, t) = 0$.
\( \Psi(L, t) = 0 \), namely

\[
\Psi(x, t) = A \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi ut}{L} \right), \quad n = 1, 2, 3 \ldots \tag{2.32}
\]

That this is the general solution can be verified by substitution of Eq. (2.32) into Eq. (2.31). The fundamental solution \((n = 1)\) and the first two harmonics \((n = 2 \text{ and } n = 3)\) are shown in Fig. 2.9. The frequencies \(\nu\) of the solutions are also discrete. The time for one cycle \(t = 1/\nu\) such that \(n\pi u t/L = 2\pi\); thus

\[
\nu = \frac{n\pi u}{2L}, \quad n = 1, 2, 3, \ldots
\]

Figure 2.9. Standing wave solutions of a vibrating string. The solution corresponding to \(n = 1\) is called the fundamental mode.

Notice that the solution of the wave equation Eq. (2.32) is separable, i.e., it has the form \(\Psi(x, t) = \psi(x)T(t) = \psi(x)\sin(2\pi \nu t)\). Substitution of this separable form into Eq. (2.31) yields

\[
\frac{d^2\psi(x)}{dx^2} + \frac{4\pi^2 \nu^2}{u^2} \psi(x) = 0
\]

or, since \(u = \lambda \nu\),

\[
\frac{d^2\psi(x)}{dx^2} + \frac{4\pi^2}{\lambda^2} \psi(x) = 0. \tag{2.33}
\]
To generalization to three-dimensions, the operator $d^2/dx^2 \rightarrow \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \equiv \nabla^2$ gives
\[
\nabla^2 \psi(x, y, z) + \frac{4\pi^2}{\lambda^2} \psi(x, y, z) = 0. \tag{2.34}
\]

Now we apply this wave equation to an electron bound to an atomic nucleus. The nucleus produces an electric field or electric force on the electron $V(x, y, z)$. The electron with (rest) mass $m$ has a total energy $E$, kinetic energy $T$, and a potential energy $V$ such that $T = E - V$. The wavelength of the electron is \( \lambda = h/p = h/\sqrt{2mT} = h/\sqrt{2m(E - V)} \) (assuming the electron is non-relativistic). Substitution for $\lambda$ into Eq. (2.34) gives
\[
- \frac{\hbar^2}{8\pi^2 m} \nabla^2 \psi(x, y, z) + V(x, y, z) \psi(x, y, z) = E \psi(x, y, z). \tag{2.35}
\]

This equation is known as the steady-state Schrödinger’s wave equation, and is the fundamental equation of quantum mechanics. This is a homogeneous equation in which everything on the left-hand side is known (except, of course, $\psi(x, y, z)$, but in which the electron energy $E$ on the right is not known. Such an equation generally has only the trivial null solution ($\psi = 0$); however, non-trivial solutions can be found if $E$ has very precise and discrete values\(^3\) $E = E_n, n = 0, 1, 2, \ldots$. This equation then says that an electron around a nucleus can have only very discrete values of $E = E_n$, a fact well verified by experiment. Moreover, the wave solution of $\psi_n(x, y, z)$ associated with a given energy level $E_n$ describes the amplitude of the electron wave. The interpretation of $\psi_n$ is discussed below.

In Addendum 2, example solutions of Schrödinger’s wave equation are presented for the interested reader.

### 2.3.2 The Wave Function

The non-trivial solution $\psi_n(x, y, z)$ of Eq. (2.35) when $E = E_n$ (an eigenvalue of the equation) is called a wave function. In general, this is a complex quantity which extends over all space, and may be thought of as the relative amplitude of a wave associated with the particle described by Eq. (2.35). Further, because Eq. (2.35) is a homogeneous equation, then, if $\psi'$ is a solution, so is $\psi = A\psi'$, where $A$ is an arbitrary constant. It is usual to choose $A$ so that the integral of $|\psi|^2$ over all space equals unity, i.e.,\(^4\)
\[
\int \int \int \psi(x, y, z)\psi^*(x, y, z) dV = 1. \tag{2.36}
\]

Just as the square of the amplitude of a classical wave defines the intensity of the wave, the square of the amplitude of the wave function $|\psi|^2$ gives the probability of

---

\(^3\)Mathematicians call such an equation (subject to appropriate boundary conditions) an eigenvalue problem in which $E_n$ is called the eigenvalue and the corresponding solution $\psi_n(x, y, z)$ the eigenfunction.

\(^4\)Here $\psi^*$ denotes the complex conjugate of $\psi$. 

---
finding the particle at any position in space. Thus, the probability that the particle is in some small volume $dV$ around the point $(x, y, z)$ is

$$\text{Prob} = |\psi(x, y, z)|^2 dV = \psi(x, y, z)\psi^*(x, y, z) dV$$

We see from this interpretation of $\psi$ that the normalization condition of Eq. (2.36) requires that the particle be somewhere in space.

### 2.3.3 The Uncertainty Principle

With quantum/wave mechanics, we see it is no longer possible to say that a particle is at a particular location; rather, we can say only that the particle has a probability $\psi\psi^* dV$ of being in a volume $dV$. It is possible to construct solutions to the wave equation such that $\psi\psi^*$ is negligibly small except in a very small region of space. Such a wave function thus localizes the particle to the very small region of space. However, such localized wave packets spread out very quickly so that the subsequent path and momentum of the particle is known only within very broad limits. This idea of there being uncertainty in a particle’s path and its speed or momentum was first considered by Heisenberg in 1927.

If one attempts to measure both a particle’s position along the $x$-axis and its momentum, there will be an uncertainty $\Delta x$ in the measured position and an uncertainty $\Delta p$ in the momentum. Heisenberg’s uncertainty principle says there is a limit to how small these uncertainties can be, namely

$$\Delta x \Delta p \geq \frac{h}{2\pi}.$$  \hspace{1cm} (2.37)

This limitation is a direct consequence of the wave properties of a particle. The uncertainty principle can be derived rigorously from Schrödinger’s wave equation; however, a more phenomenological approach is to consider an attempt to measure the location of an electron with very high accuracy. Conceptually, one could use an idealized microscope, which can focus very short wavelength light to resolve points that are about $10^{-11}$ m apart. To “see” the electron a photon must scatter from it and enter the microscope. The more accurately the position is to be determined (i.e., the smaller $\Delta x$), the smaller must be the light’s wavelength (and the greater the photon’s energy and momentum). Consequently, the greater is the uncertainty in the electron’s momentum $\Delta p$ since a higher energy photon, upon rebounding from the electron, will change the electron’s momentum even more. By observing a system, the system is necessarily altered.

The limitation on the accuracies with which both position and momentum (speed) can be known is an important consideration only for systems of atomic dimensions. For example, to locate a mass of 1 g to within 0.1 mm, the minimum uncertainty in the mass’s speed, as specified by Eq. (2.37), is about $10^{-26}$ m/s, far smaller than errors introduced by practical instrumentation. However, at the atomic and nuclear levels, the uncertainty principle provides a very severe restriction on how position and speed of a particle are fundamentally intertwined.

There is a second uncertainty principle (also by Heisenberg) relating the uncertainty $\Delta E$ in a particle’s energy $E$ and the uncertainty in the time $\Delta t$ at which the
particle had the energy, namely
\[
\Delta E \Delta t \geq \frac{\hbar}{2\pi}.
\] (2.38)

This restriction on the accuracy of energy and time measurements is a consequence of the time-dependent form of Schrödinger’s wave equation (not presented here), and is of practical importance only in the atomic world. In the atomic and subatomic world involving transitions between different energy states, energy need not be rigorously conserved during very short time intervals \(\Delta t\), provided the amount of energy violation \(\Delta E\) is limited to \(\Delta E \approx \hbar/(2\pi \Delta t)\). This uncertainty principle is an important relation used to estimate the lifetimes of excited nuclear states.

### 2.3.4 Success of Quantum Mechanics

Quantum mechanics has been an extremely powerful tool for describing the energy levels and the distributions of atomic electrons around a nucleus. Each energy level and configuration is uniquely defined by four quantum numbers: \(n\) the principal quantum number, \(l\) the orbital angular momentum quantum number, \(m_l\) the \(z\)-component of the angular momentum, and \(m_s = \pm 1/2\), the electron spin number. These numbers arise naturally from the analytical solution of the wave equation (as modified by Dirac to include special relativity effects) and thus avoid the ad hoc introduction of orbital quantum numbers required in earlier atomic models.

Inside the nucleus, quantum mechanics is also thought to govern. However, the nuclear forces holding the neutrons and protons together are much more complicated than the electromagnetic forces binding electrons to the nucleus. Consequently, much work continues in the application of quantum mechanics (and its more general successor quantum electrodynamics) to predicting energy and configuration states of nucleons. Nonetheless, the fact that electronic energy levels of an atom and nuclear excited states are discrete with very specific configurations is a key concept in modern physics. Moreover, when one state changes spontaneously to another state, energy is emitted or absorbed in specific discrete amounts.

### 2.4 Addendum 1: Derivation of Some Special Relativity Results

In this addendum, the relativistic effects for time dilation, length contraction, and mass increase are derived.

#### 2.4.1 Time Dilation

Consider a timing device that emits a pulse of light from a source and then records the time the light takes to travel a distance \(d\) to a detector. In a stationary frame of reference (left-hand figure of Fig. 2.10), the travel time is denoted by \(t_o\) (the proper time) and the separation distance is given by \(d = ct_o\).

Now observe the timing device as it moves to the right at a steady speed \(v\) relative to a stationary observer (right-hand figure of Fig. 2.10). During the time \(t\) it takes the pulse to appear to travel from the source to the detector, the detector has moved a distance \(ct\) to the right. From Einstein’s first postulate (\(c\) is the same to all observers), the total distance the photon appears to travel is \(ct\). The distance between source and detector is still \(d = ct_o\). From the right-hand figure, these three
Figure 2.10. A light pulse leaves a source (S) and travels to a detector (D) in a stationary system (left) and in a system moving to the right with a uniform speed v (right).

d = ct

Figure 2.11. Left figure: a rod moves past a stationary measurement pointer at a steady speed v. Right figure: in the system moving with the rod, the pointer appears to be moving backward with speed v.

2.4.2 Length Contraction

Consider a rod of length \( L_0 \) when stationary (its proper length). If this rod is allowed to move to the right with constant speed \( v \) (see left figure in Fig. 2.11), we can measure its apparent length by the time \( t_0 \) it takes to move past some stationary reference pointer. This is a proper time, since the transit time has been made at the same position in the observer’s frame of reference. Thus the rod length appears to be

\( L = vt_0. \) (2.40)

Now consider the same measurement made by an observer moving with the rod (see the right-hand figure in Fig. 2.11). In this frame of reference, the reference

\[ c^2 t^2 = v^2 t'^2 + c^2 t_0^2, \]

from which it follows

\[ t = \frac{t_0}{\sqrt{1 - v^2/c^2}}. \] (2.39)

Thus we see that the travel time for the pulse of light appears to lengthen or dilate as the timing device moves faster with respect to a stationary observer. In other words, a clock in a moving frame of reference (relative to some observer) appears to run more slowly. This effect (like all other special relativity effects) is reciprocal. To the moving clock, the stationary observer’s watch appears to be running more slowly.
pointer appears to be moving to the left with speed $v$. The rod has length $L_0$ to
this observer, so the time it takes for the reference pointer to move along the rod is
$t = L_0/v$. This is a dilated time since the reference pointer is moving with respect
to the observer. Substitution of Eq. (2.39), gives $L_0/v = t = t_0/\sqrt{1-v^2/c^2}$, or
\[ t_0 = \frac{L_0}{v}\sqrt{1-v^2/c^2}. \]

Finally, substitution of this result into Eq. (2.40) shows the proper length $L_0$ is
related to the rod’s apparent length $L$ by
\[ L = L_0\sqrt{1-v^2/c^2}. \]

Thus, we see that the width of an object appears to decrease or contract as it moves
at a uniform speed past an observer.

2.4.3 Mass Increase

By considering an elastic collision between two bodies, we can infer that the mass
of a body varies with its speed. Suppose there are two experimenters, one at rest in
system $S$ and the other at rest in system $S'$, which is moving with speed $v$ relative to
$S$ along the $x$-direction. Both experimenters, as they pass, launch identical elastic
spheres with speed $u$ (as judged by each experimenter) in a direction perpendicular
to the $x$-axis (again, as judged by each), so that a head-on collision occurs midway
between them. Each sphere has rest mass $m_o$. For a launch speed $u \ll c$, then each
experimenter’s sphere has mass $m_o$ in his frame of reference. Fig. 2.12 illustrates
the trajectories of the spheres that each experimenter observes.

![Diagram](image)

**Figure 2.12.** The trajectories of two colliding elastic spheres as seen by the experimenter
in the stationary $S$ system (left-hand figure), and by the experimenter in the $S'$ system
(right-hand figure).

To the experimenter in $S$ (left-hand figure of Fig. 2.12), the time for his sphere to
travel to the collision point is $t_o = h/u$. However, his colleague’s sphere appears to
take longer to reach the collision, namely the dilated time $t = t_0/\sqrt{1-v^2/c^2}$. Thus
to the $S$ experimenter, the sphere from $S'$ appears to be moving in the negative
$y$-direction with a speed
\[ w = \frac{h}{t} = \frac{h}{t_0}\sqrt{1-v^2/c^2} = u\sqrt{1-v^2/c^2}, \]

where we have used Eq. (2.39).
By Einstein's second postulate (physical laws are the same in all inertial frames of reference), momentum must be conserved in the collision of the spheres. To conserve momentum in this experiment, we require that the total momentum of the two spheres in the y-direction be the same before and after the collision, i.e.,

\[ m_0u - mw = -m_0u + mw, \]

or using Eq. (2.42)

\[ m_0u = mw = mu\sqrt{1 - \frac{v^2}{c^2}}. \]

This reduces to

\[ m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (2.43) \]

### 2.5 Addendum 2: Solutions to Schrödinger's Wave Equation

#### 2.5.1 The Particle in a Box

The solution of the Schrödinger wave equation is generally very difficult and usually requires approximations or sophisticated numerical techniques. However, there are several simple problems for which analytical solutions can be obtained. These exact solutions can be used to show clearly the differences between quantum and classical mechanics. In addition, many of these simple problems can be used to model important phenomena such as electron conduction in metals and radioactive alpha decay.

One such problem is the description of the motion of a free particle of mass \( m \) confined in a "one-dimensional" box. Within the box, \( 0 < x < a \), the particle is free to move. However, it is not allowed past the walls at \( x = 0 \) and \( x = a \), and if the particle reaches a wall it is reflected elastically back into the interior of the box.

In a classical description, the particle can move back and forth between the bounding walls with any kinetic energy \( E \) or speed \( v \). Moreover, the probability of finding the particle in any differential width \( dx \) between the walls is the same for all \( dx \).

In a quantum mechanical description of the particle, given by the Schrödinger wave equation, the potential energy or force on the particle \( V(x) = 0 \) inside the box, and infinite outside. Since the potential energy is infinite outside the box, the particle cannot exist outside the box, and the wave functions \( \psi(x) \) must vanish. The Schrödinger equation, Eq. (2.35), for the particle is,

\[ \frac{d^2\psi}{dx^2} + \frac{8\pi^2mE}{\hbar^2}\psi = 0, \quad 0 < x < a. \quad (2.44) \]

If we let \( k^2 = \frac{8\pi^2mE}{\hbar^2} \), the general solution of this equation is,

\[ \psi(x) = A \sin kx + B \cos kx, \quad (2.45) \]

where \( A \) and \( B \) are arbitrary constants. This solution must also satisfy the boundary conditions, namely, \( \psi(0) = 0 \) and \( \psi(a) = 0 \). These boundary conditions, as shown below, severely restrict the allowed values of not only \( A \) and \( B \) but also of \( k \).

Application of the boundary condition \( \psi(0) = 0 \) at the left wall to Eq. (2.45) yields

\[ \psi(0) = 0 = A \sin(0) + B \cos(0) = B \quad (2.46) \]
which forces us to set $B = 0$. Thus, the general solution reduces to $\psi(x) = A \sin kx$. Application of the boundary condition $\psi(a) = 0$ at the right wall produces the restriction that

$$\psi(a) = A \sin ka = 0. \quad (2.47)$$

Clearly, we could satisfy this restriction by taking $A = 0$. But this would say that the wave function $\psi(x) = 0$ everywhere in the box. This trivial null solution is unrealistic since it implies the particle cannot be in the box. However, we can satisfy Eq. (2.47) with $A \neq 0$ if we require $\sin ka = 0$. Since the sine function vanishes at all multiples of $\pi$, we see that $k$ must be chosen as

$$k = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots \quad (2.48)$$

These discrete or quantized values of $k$ are denoted by $k_n$.

Thus, the only allowed solutions of the Schrödinger wave equation are the functions

$$\psi_n(x) = A \sin \frac{n\pi x}{a}, \quad n = 1, 2, 3, \ldots \quad (2.49)$$

where $A$ is any non-zero constant. Moreover, the particle’s energy, $E = \hbar^2 k/(8\pi^2 m)$, can have only discrete values given by,

$$E_n = \frac{k_n \hbar^2}{8\pi^2 m} = \frac{n^2 \hbar^2}{8ma^2}, \quad n = 1, 2, 3, \ldots \quad (2.50)$$

These discrete values $E_n$ are sometimes called the eigenvalues of Eq. (2.44) and $n$ is a quantum number. Thus, in a quantum mechanical description of the particle, the particle’s kinetic energy (and hence its speed and momentum) must have very discrete or quantized values. By contrast, a classical description allows the particle to have any kinetic energy or speed.

The wave function associated with the particle with energy $E_n$ (the so-called eigenfunction of Eq. (2.44)) is given by Eq. (2.49). The constant $A$ in Eq. (2.49) is chosen to normalize the square of the wave function to unity, i.e.,

$$\int_0^a |\psi_n(x)|^2 dx = A^2 \int_0^a \sin^2 \frac{n\pi x}{a} dx = 1. \quad (2.51)$$

To satisfy this normalization, $A = \sqrt{2/a}$ and the normalized wave functions become,

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}, \quad n = 1, 2, 3, \ldots \quad (2.52)$$

This normalization condition of Eq. (2.51) is required to give a probability of unity that the particle is somewhere in the box (see Section 2.3.2).

The probability that a particle, in a given energy state $E_n$, occupies any given volume element $dx$ of the one-dimensional box is given by the quantum mechanical probability distribution, namely

$$|\psi(x)|^2 dx = \frac{2}{a} \sin^2 \left(\frac{n\pi x}{a}\right) dx, \quad (2.53)$$
which is a wave-like function and is plotted in Fig. 2.13 for the quantum state \( n = 3 \). Because the square of the wave function is the probability of occupancy in a unit volume element about \( x \), we see that the probability of finding the particle at any particular location varies with the location. This probability distribution \( |\psi_3(x)|^2 \) vanishes not only at the wall \( (x = 0 \text{ and } x = a) \), but also at \( x = a/3 \) and \( x = 2a/3 \). By contrast it has maxima at \( x = a/6, x = a/2, \) and \( x = 5a/6 \). This quantum mechanical result for where the particles is likely to be in the box is significantly different than the classical result for which the probability of occupancy is the same for any volume element \( dx \). The differences are shown in Fig. 2.13.

\[ \text{Figure 2.13. The probability distribution for finding a particle in quantum state } n = 3 \text{ with energy } E_3 \text{ at points in a one-dimensional box.} \]

The quantum mechanical standing-wave distribution for the location of an electron trapped in a box can be seen in the lower-right figure of [Fig. 1.1]. Here the circle of iron atoms acts like the walls of our box, and the valence electrons, normally free to move around the surface of the copper substrate, are now confined. A standing wave for the location of these trapped electrons is clearly seen. This probability wave is completely analogous to the 1-dimensional analysis above.

If the particle is confined to a three-dimensional box with edges of length \( a, b, c \), the normalized wave functions for the stationary states and the values for the energy levels are found to be,

\[ \psi_{n_1, n_2, n_3}(x, y, z) = 2\sqrt{\frac{2}{abc}} \sin\left(\frac{n_1\pi x}{a}\right) \sin\left(\frac{n_2\pi y}{b}\right) \sin\left(\frac{n_3\pi z}{c}\right) \]

and

\[ E_{n_1, n_2, n_3} = \frac{\hbar^2}{8m} \left[ \frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2} \right], \]

where \( n_1, n_2, \) and \( n_3 \) denote a set of positive integers.

### 2.5.2 The Hydrogen Atom

The hydrogen atom, the simplest atom, can be considered as a system of two interacting point charges, the proton (nucleus) and an electron. The electrostatic
attraction between the electron and proton is described by Coulomb's law. The potential energy of a bound electron is given by,

\[ V(r) = -\frac{e^2}{r} \]  

(2.56)

where \( r \) is the distance between the electron and the proton.

The three-dimensional Schrödinger wave equation in spherical coordinates for the electron bound to the proton is,

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{8\pi^2 \mu}{\hbar^2} [E - V(r)] \psi = 0, \]  

(2.57)

where \( \psi = \psi(r, \theta, \phi) \) and \( \mu \) is the electron mass (more correctly the reduced mass of the system).

To solve this partial differential equation for the wave function \( \psi(r, \theta, \phi) \), we first replace it by three equivalent ordinary differential equations involving functions of only a single independent variable. To this end, we use the "separation of variables" method, and seek a solution of the form

\[ \psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi). \]  

(2.58)

Substitute this form into Eq. (2.57) and multiply the result by \( r^2 \sin \theta / (R\Theta \Psi) \) to obtain

\[ \frac{\sin^2 \theta}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{1}{\Phi(\phi)} \frac{d^2 \Phi(\phi)}{d\phi^2} + \frac{\sin \theta}{\Theta(\theta)} \frac{d^2 \Theta(\theta)}{d\theta^2} + \frac{8\pi^2 \mu}{\hbar^2} r^2 \sin^2 \theta [E - V(r)] = 0. \]  

(2.59)

The second term is only a function of \( \phi \) while the other terms are independent of \( \phi \). This term, therefore, must equal a constant, \(-m^2 \) say. Thus,

\[ \frac{d^2 \Phi(\phi)}{d\phi^2} = -m^2 \Phi(\phi). \]  

(2.60)

Equation 2.59, upon division by \( \sin^2 \theta \) and rearrangement becomes

\[ \frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) + \frac{8\pi^2 \mu r^2}{\hbar^2} [E - V(r)] = -\frac{1}{\sin \theta \Theta(\theta)} \frac{d^2 \Theta(\theta)}{d\theta^2} + \frac{m^2}{\sin^2 \theta} = 0. \]  

(2.61)

Since the terms on the left are functions only of \( r \) and the terms on the right are functions only of \( \theta \), both sides of this equation must be equal to the same constant, \( \beta \) say. Thus we obtain two ordinary differential equations, one for \( R(r) \) and one for \( \Theta(\theta) \), namely

\[ \frac{1}{\sin \theta} \frac{d^2 \Theta(\theta)}{d\theta^2} - \frac{m^2}{\sin^2 \theta} \Theta(\theta) + \beta \Theta(\theta) = 0, \]  

(2.62)

and

\[ \frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{dR(r)}{dr} \right] + \frac{\beta}{r^2} R(r) + \frac{8\pi^2 \mu}{\hbar^2} [E - V(r)] R(r) = 0. \]  

(2.63)

We now have three ordinary, homogeneous, differential equations [Eq. (2.60), Eq. (2.62), and Eq. (2.63)], whose solutions, when combined, given the entire wave
function $\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$. Each of these equations is an eigenvalue problem which yields a "quantum number." These quantum numbers are used to describe the possible electron configurations in a hydrogen atom. The solutions also yield relationships between the quantum numbers. We omit the solution details – see any book on quantum mechanics for the explicit solutions. We restrict our discussion to the essential features than arise from each equation.

The most general solution of Eq. (2.60) is

$$
\Phi(\phi) = A\sin m\phi + B\cos m\phi,
$$

where $A$ and $B$ are arbitrary. However, we require $\Phi(0) = \Phi(2\pi)$ since $\phi = 0$ and $\phi = 2\pi$ are the same azimuthal angle. This boundary condition then requires $m$ to be an integer, i.e., $m = 0, \pm 1, \pm 2, \ldots$. For each azimuthal quantum number $m$, the corresponding solution is denoted by $\Phi_m(\phi)$.

Equation (2.62) in $\theta$ has normalizable solutions only if the separation constant has the form $\ell(\ell + 1)$ where $\ell$ is a positive integer or zero, and is called the angular momentum quantum number. Moreover, the azimuthal quantum number $m$ must be restricted to $2\ell + 1$ integer values, namely $m = 0, \pm 1, \pm 2, \ldots, \pm \ell$. The corresponding solutions of Eq. (2.62) are denoted by $\Theta_{\ell m}(\theta)$ and are known to mathematicians as the associate Legendre functions of the first kind; however, these details need not concern us here.

Finally, the solution of Eq. (2.63) for the radial component of the wave function, with $\beta = \ell(\ell + 1)$ and $V(r) = -e^2/r$, has normalizable solutions only if the electron’s energy has the (eigen)value

$$
E_n = -\frac{2\pi^2\mu e^2}{\hbar^2 n^2}, \quad \text{where } n = 1, 2, 3, \ldots
$$

The integer $n$ is called the principal quantum number. Moreover, to obtain a solution the angular quantum number $\ell$ must be no greater than $n - 1$, i.e., $\ell = 0, 1, 2, \ldots, (n - 1)$. The corresponding radial solution is denoted by $R_{n\ell}(r)$ and mathematically is related to the associated Laguerre function.

Thus, the wave functions for the electron bound in the hydrogen atom have only very discrete forms $\psi_{n\ell m}(r, \theta, \phi) = R_{n\ell}(r)\Theta_{\ell m}(\theta)\Phi_m(\phi)$. For the hydrogen atom, the energy of the electron is given by Eq. (2.65) and is independent of the angular or azimuthal quantum numbers $m$ and $\ell$ (this is not true for multielectron atoms).

### Special Notation for Electron States

A widely used, but strange, notation has been of long standing in describing the $n$ and $\ell$ quantum numbers of particular electron states. The letters $s, p, d, f, g, h$ and $i$ are used to denote values of the angular momentum quantum number $\ell$ of 0, 1, 2, 3, 4, 5 and 6, respectively. The value of $n$ is then used as a prefix to the angular moment letter. Thus a bound electron designated as $5f$ refers to an electron with $n = 5$ and $\ell = 3$.

### Examples of Wave Functions for Hydrogen

In Fig. 2.14, density plots of $|\psi_{n\ell m}|^2 = \psi_{n\ell m}^*\psi_{n\ell m}$ are shown for different electron states in the hydrogen atom. These plots are slices through the three-dimensional
Figure 2.14. Density plots of $\psi^*\psi$ for different electron eigenstates in the hydrogen atom. Plots are sectional views of the probability density in a plane containing the polar axis, which is vertical and in the plane of the paper. Scale of the right group is about 15% bigger than that of the left group. After: R.B. Leighton, Principles of Modern Physics, McGraw Hill, New York, 1959.
\( |\psi|^2 \) in a plane perpendicular to the x-axis and through the atom's center. Since \( |\psi|^2 \) is the probability of finding the electron in a unit volume, the density plots directly show the regions where the electron is most likely to be found. The s states (\( \ell = 0 \)) are spherically symmetric about the nucleus, while all the others have azimuthal and/or polar angle dependence.

**Electron Energy Levels in Hydrogen**

For hydrogen, the energy of the bound electron is a function of \( n \) only (see Eq. (2.65)). Thus, for \( n > 1 \), the quantum numbers \( \ell \) and \( m \) may take various values without changing the electron binding energy, i.e., the allowed electron configurations fall into sets in which all members of the set have the same energy. Thus, in Fig. 2.14, an electron has the same energy in any of the four 4f electron states or the three 4d states, even though the distribution of the electron wave function around the nucleus is quite different for each state. Such states with the same electron binding energy are said to be degenerate.

**The Spin Quantum Number**

It was found, first from experiment and later by theory, that each quantum state (specified by values for \( n, \ell \) and \( m \)) can accommodate an electron in either of two spin orientations. The electron, like the proton and neutron, has an inherent angular momentum with a value of \( \frac{1}{2} \hbar/(2\pi) \). In a bound state (defined by \( n, \ell \) and \( m \)) the electron can have its spin “up” or “down” with respect to the z-axis used to define angular momentum. Thus a fourth quantum number \( m_s = \pm \frac{1}{2} \) is needed to unambiguously define each possible electron configuration in an atom.

Dirac showed in 1928 that when the Schrödinger wave equation is rewritten to include relativistic effects, the spin quantum number \( m_s \) is inherent in the solution along with the quantum numbers \( n, \ell \), and \( m \), which were also inherent in the wave function solution of the non-relativistic Schrödinger's wave equation.

### 2.5.3 Energy Levels for Multielectron Atoms

The solution of the Schrödinger's equation for the hydrogen atom can be obtained analytically, but the solution for a multielectron system cannot. This difficulty arises because of the need to add a repulsive component to the potential energy term to account for the interactions among the electrons. To obtain a solution for a multielectron atom, numerical approximation techniques must be used together with high speed computers. Such a discussion is far beyond the scope of this text.

There is, however, one important result for multielectron atoms that should be described here. The energies of the electronic levels in atoms with more than one electron are functions of both \( n \) and \( \ell \). The energy degeneracy in the angular momentum quantum number disappears. Thus states with the same \( n \) but different \( \ell \) values have slightly different energies, and there is a significant reordering of the electron energy levels compared to those in the hydrogen atom.

Electron energy levels with the same value of \( n \) and \( \ell \) but different values of \( m \) are still degenerate in energy; however, this degeneracy is removed in the presence of a strong external magnetic field (the Zeeman effect). An electron moving about a nucleus creates a magnetic dipole whose strength of interaction with an external magnetic field varies with the quantum numbers \( \ell \) and \( m \).
Table 2.2. Electron shell arrangement for the lightest elements.

<table>
<thead>
<tr>
<th>Element</th>
<th>Z</th>
<th>shell and electron designation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(K)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1s</td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>He</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Li</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Be</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>N</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>O</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>F</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>Ne</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>Na</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>Mg</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>Al</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>Si</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>P</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>Cl</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td>Ar</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>K</td>
<td>19</td>
<td>1</td>
</tr>
<tr>
<td>Ca</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>Sc</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>Ti</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>Cr</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>Mn</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Fe</td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>Co</td>
<td>27</td>
<td></td>
</tr>
<tr>
<td>Ni</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>Cu</td>
<td>29</td>
<td></td>
</tr>
<tr>
<td>Zn</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Ga</td>
<td>31</td>
<td></td>
</tr>
<tr>
<td>Ge</td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>As</td>
<td>33</td>
<td></td>
</tr>
<tr>
<td>Se</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>Br</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td>Kr</td>
<td>36</td>
<td></td>
</tr>
</tbody>
</table>

The Electronic Structure of Atoms

Each electron in an atom can be characterized by its four quantum numbers \( n, \ell, m, \) and \( m_s \). Further, according to Pauli's exclusion principle (1925), no two electrons in an atom can have the same quantum numbers. An assignment of a set of four quantum numbers to each electron in an atom, no sets being alike in all four numbers, then defines a quantum state for an atom as a whole. For ground-state atoms, the electrons are in the lowest energy electron states. For an atom in an excited state, one or more electrons are in electron states with energies higher than some vacant states. Electrons in excited states generally drop very rapidly (~ \( 10^{-7} \) s) into vacant lower energy states. During these spontaneous transitions, the difference in energy levels between the two states must be emitted as a photon (fluorescence or x rays) or be absorbed by other electrons in the atom (thereby causing them to change their energy states).

Crucial to the chemistry of atoms is the arrangement of atomic electrons into various electron shells. All electrons with the same \( n \) number constitute an electron shell. For \( n = 1, 2, \ldots, 7 \), the shells are designated \( K, L, M, \ldots, Q \).

Consider electrons with \( n = 1 \) (K shell). Since \( \ell = 0 \) (s state electrons), then \( m = 0 \) and \( m_s = \pm \frac{1}{2} \). Hence, there are only 2 Is electrons, written as \( 1s^2 \), in the K shell. In the L shell (\( n = 2 \)), \( \ell = 0 \) or 1. For \( \ell = 0 \) there are two 2s electrons (denoted by \( 2s^2 \)), and for \( \ell = 1 \) (\( m = -1, 0, 1 \)) there are six 2p electrons (denoted by \( 2p^6 \)). Thus in the L shell there are a total of eight electrons (\( 2s^22p^6 \)). Electrons with the same value of \( \ell \) (and \( n \)) are referred as a subshell. For a given subshell there are \( (2\ell + 1) \) \( m \) values, each with two \( m_s \) values, giving a total of \( 2(2\ell + 1) \) electrons per subshell, and \( 2n^2 \) electrons per shell. A shell or subshell containing the maximum number of electrons is said to be closed.

The Period Table of Elements (see Appendix A.2) can be described in terms of the possible number of electrons in the various subshells. The number of electrons in an atom equals its atomic number \( Z \) and determines its position in the Periodic Table. The chemical properties are determined by the number and arrangement of the electrons. Each element in the table is formed by adding one electron to that of the preceding element in the Periodic Table in such a way that the electron is most tightly bound to the atom. The arrangement of the electrons for the elements with electrons in only the first four shells is shown in Table 2.2.

BIBLIOGRAPHY

KAPLAN, I., Nuclear Physics, Addison-Wesley, Reading, MA, 1963.
**PROBLEMS**

1. An accelerator increases the total energy of electrons uniformly to 10 GeV over a 3000 m path. That means that at 30 m, 300 m, and 3000 m, the kinetic energy is $10^8$, $10^9$, and $10^{10}$ eV, respectively. At each of these distances, compute the velocity, relative to light ($v/c$), and the mass in atomic mass units.

2. Consider a fast moving particle whose relativistic mass $m$ is 100$\%$ percent greater than its rest mass $m_0$, i.e., $m = m_0(1 + \epsilon)$. (a) Show that the particle's speed $v$, relative to that of light, is

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(1 + \epsilon)^2}}.$$

(b) For $v/c << 1$, show that this exact result reduces to $v/c \approx \sqrt{2\epsilon}$.

3. In fission reactors one deals with neutrons having kinetic energies as high as 10 MeV. How much error is incurred in computing the speed of 10-MeV neutrons by using the classical expression rather than the relativistic expression for kinetic energy?

4. What speed (m s$^{-1}$) and kinetic energy (MeV) would a neutron have if its relativistic mass were 10% greater than its rest mass?

5. In the Relativistic Heavy Ion Collider, nuclei of gold are accelerated to speeds of 99.95% the speed of light. These nuclei are almost spherical when at rest; however, as they move past the experimenters they appear considerably flattened in the direction of motion because of relativistic effects. Calculate the apparent diameter of such a gold nucleus in its direction of motion relative to that perpendicular to the motion.

6. Muons are subatomic particles that have the negative charge of an electron but are 206.77 times more massive. They are produced high in the atmosphere by cosmic rays colliding with nuclei of oxygen or nitrogen, and muons are the dominant cosmic-ray contribution to background radiation at the earth's surface. A muon, however, rapidly decays into an energetic electron, existing, from its point of view, for only 2.20 $\mu$s, on the average. Cosmic-ray generated muons typically have speeds of about 0.998$c$ and thus should travel only a few hundred meters in air before decaying. Yet muons travel through several kilometers of air to reach the earth's surface. Using the results of special relativity explain how this is possible. HINT: consider the atmospheric travel distance as it appears to a muon, and the muon lifetime as it appears to an observer on the earth's surface.


8. At what energy (in MeV) can a photon lose at most one-half of its energy in Compton scattering?
9. A 1 MeV photon is Compton scattered at an angle of 55 degrees. Calculate (a) the energy of the scattered photon, (b) the change in wavelength, and (c) the recoil energy of the electron.

10. Show that the de Broglie wavelength of a particle with kinetic energy \( T \) can be written as

\[
\lambda = \frac{h}{\sqrt{m_o \sqrt{T}}} \left[ 1 + \frac{m}{m_o} \right]^{-1/2}
\]

where \( m_o \) is the particle's rest mass and \( m \) is its relativistic mass.

11. Apply the result of the previous problem to an electron. (a) Show that when the electron's kinetic energy is expressed in units of eV, its de Broglie wavelength can be written as

\[
\lambda = \frac{17.35 \times 10^{-8}}{\sqrt{T}} \left[ 1 + \frac{m}{m_o} \right]^{-1/2} \text{ cm.}
\]

(b) For non-relativistic electrons, i.e., \( m \approx m_o \), show that this result reduces to

\[
\lambda = \frac{12.27 \times 10^{-8}}{\sqrt{T}} \text{ cm.}
\]

(c) For very relativistic electrons, i.e., \( m \gg m_o \), show that the de Broglie wavelength is given by

\[
\lambda = \frac{17.35 \times 10^{-8}}{\sqrt{T}} \sqrt{\frac{m_o}{m}} \text{ cm.}
\]

12. What are the wavelengths of electrons with kinetic energies of (a) 10 eV, (b) 1000 eV, and (c) \( 10^7 \) eV?

13. What is the de Broglie wavelength of a water molecule moving at a speed of 2400 m/s? What is the wavelength of a 3-g bullet moving at 400 m/s?

14. If a neutron is confined somewhere inside a nucleus of characteristic dimension \( \Delta x \approx 10^{-14} \) m, what is the uncertainty in its momentum \( \Delta p \)? For a neutron with momentum equal to \( \Delta p \), what is its total energy and its kinetic energy in MeV? Verify that classical expressions for momentum and kinetic energy may be used.

15. Repeat the previous problem for an electron trapped in the nucleus. HINT: relativistic expressions for momentum and kinetic energy must be used.