Problem: An accelerator increases the total energy of electrons uniformly to 10 GeV over a 3000 m path. That means that at 30 m, 300 m, and 3000 m, the kinetic energy is $10^8$, $10^9$, and $10^{10}$ eV, respectively. At each of these distances, compute the velocity, relative to light ($v/c$), and the mass in atomic mass units.

Solution: The kinetic energy $T$ is equal to the difference between the total energy ($mc^2$) and the particle's (i.e., the electron's) rest energy as given by Eq. (2.10), $T = mc^2 - m_ec^2$. Next, Eq. (2.2) can be rewritten to determine the velocity $v$ relative to the speed of light $c$ by solving for $v/c$.

**Evaluation at 30 m:**

$$E_{T30} := \frac{10^8}{10^6} \text{ MeV, Total energy of electron at 30 meters}$$

$$c := 2.99792458 \times 10^8 \text{ m/s, Speed of light}$$

$$m_{30} := \frac{E_{T30}}{c^2} \frac{c^2}{931.494013} \text{ amu, Mass of electron at 30 m}$$

$$m_{30} = 0.1074$$

$$m_e := 5.48579911 \times 10^{-4} \text{ amu, Rest mass of an electron (Table 1.5)}$$

$$v_{c30} := \sqrt{1 - \left( \frac{m_e}{m_{30}} \right)^2} \text{ dimensionless, Ratio of v/c at 30 meters}$$

$$v_{c30} = 9.9998694392 \times 10^{-1}$$

**Evaluation at 300 m:**

$$E_{T300} := \frac{10^9}{10^6} \text{ MeV, Total energy of electron at 300 meters}$$
\[ c := 2.99792458 \times 10^8 \text{ m/s, Speed of light} \]

\[ m_{300} := \frac{ET_{300}}{c^2} \times \frac{c^2}{931.494013} \text{ amu, Mass of electron at 300 m} \]

\[ m_{300} = 1.0735 \]

\[ m_e := 5.48579911 \times 10^{-4} \text{ amu, Rest mass of an electron (Table 1.5)} \]

\[ v_{c300} := \sqrt{1 - \left( \frac{m_e}{m_{300}} \right)^2} \text{ dimensionless, Ratio of v/c at 300 meters} \]

\[ v_{c300} = 9.9999986944 \times 10^{-1} \]

**Evaluation at 3000 m:**

\[ ET_{3000} := \frac{10^{10}}{10^6} \text{ MeV, Total energy of electron at 3000 meters} \]

\[ c := 2.99792458 \times 10^8 \text{ m/s, Speed of light} \]

\[ m_{3000} := \frac{ET_{3000}}{c^2} \times \frac{c^2}{931.494013} \text{ amu, Mass of electron at 3000 m} \]

\[ m_{3000} = 10.7354 \]

\[ m_e := 5.48579911 \times 10^{-4} \text{ amu, Rest mass of an electron (Table 1.5)} \]

\[ v_{c3000} := \sqrt{1 - \left( \frac{m_e}{m_{3000}} \right)^2} \text{ dimensionless, Ratio of v/c at 30 meters} \]
This concludes the problem solution for Homework Problem 2.1!

[HW_2_1.xmcd]
Problem: Consider a fast moving particle whose relativistic mass \( m \) is \( 100\epsilon \) percent greater than its rest mass \( m_o \), i.e., \( m = m_o(1+\epsilon) \).

(a) Show that the particle's speed \( v \), relative to that of light is \( \frac{v}{c} = (1 - \frac{1}{1 + \epsilon^2})^{0.5} \)

(b) For \( \frac{v}{c} < < 1 \), show that this exact result reduces to \( \frac{v}{c} \) is approximately \( (2\epsilon)^{0.5} \)

Solution: The solution using Eq. (2.2) is as follows when we let \( \frac{v}{c} = \frac{v}{c} \):

\[
m = m_o(1 + \epsilon)
\]

From the problem definition

\[
m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Relativistic mass with respect to the rest mass from Eq. (2.2)

\[
m = \frac{m_o}{\sqrt{1 - \frac{v}{c}^2}}
\]

\[
m_o(1 + \epsilon) = \frac{m_o}{\left(1 - \frac{v}{c}^2\right)^{0.5}}
\]

\[
\frac{v}{c} = \left(\frac{2\epsilon + \epsilon^2}{1 + \epsilon}\right) = \sqrt{\frac{2\epsilon + \epsilon^2}{(1 + \epsilon)^2}} = \sqrt{\frac{2\epsilon + \epsilon^2 + 1 - 1}{(1 + \epsilon)^2}} = \sqrt{\frac{(1 + \epsilon)^2 - 1}{(1 + \epsilon)^2}}
\]

\[
\frac{v}{c} = \sqrt{1 - \frac{1}{(1 + \epsilon)^2}}
\]
Now if $\frac{v}{c}$ is very small, then utilizing the result above and the Taylor Series expansion, Eq. (2.2) becomes:

$$vvcc = \sqrt{1 - \frac{1}{(1+\varepsilon)^2}} = \sqrt{1 - \frac{1}{(1+2\varepsilon)}} = \sqrt{\frac{1+2\varepsilon}{(1+2\varepsilon)} - \frac{1}{(1+2\varepsilon)}} = \sqrt{\frac{(1+2\varepsilon-1)}{(1+2\varepsilon)}}$$

Note that $1+\varepsilon$ is approximately equal to 1 when $\frac{v}{c}$ is very small!

$$vvcc = \sqrt{\frac{2\varepsilon}{1}} = \sqrt{2\varepsilon}$$

This concludes the problem solution for Homework Problem 2.2!
Problem: In fission reactors, one deals with neutrons having kinetic energies as high as 10 MeV. How much error is incurred in computing the speed of 10 MeV neutrons by using the classical expression rather than the relativistic expression for kinetic energy?

Solution: We must determine $v$, the velocity of the neutrons by both classical and relativistic expressions. The solution using Eq. (2.10) for the relativistic expression for kinetic energy is as follows:

$m_n := 1.0086649233$ amu, Rest mass of neutron

$T := 10$ MeV, Kinetic energy of neutrons

**Classical Mechanics**

\[ T = \frac{1}{2} m_n v_{\text{classical}}^2 \]

\[ v_{\text{classical}} := \sqrt{\frac{2 \cdot T}{m_n}} \quad \text{(MeV/amu)}^{1/2}, \text{Speed of neutrons in classical mechanics approach} \]

\[ v_{\text{classical}} = 4.4529 \]

**Relativistic Mechanics**

\[ T = m_{\text{rel}} c^2 - m_n c^2 = m_n c^2 \left( \sqrt{1 - \frac{v_{\text{rel}}^2}{c^2}} - 1 \right) \]

\[ \frac{T}{(m_n c^2)} + 1 = \sqrt{\frac{1}{1 - \frac{v_{\text{rel}}^2}{c^2}}} \]

$c := \sqrt{931.49}$ (MeV/amu)$^{1/2}$, Energy-to-mass conversion factor for the speed of light
\[ v_{\text{rel}} := \left( 2 \cdot T \cdot m_n \cdot c^2 + T^2 \right)^{\frac{1}{2}} \cdot \frac{c}{T + m_n \cdot c^2} \]  

(MeV/amu)\(^{1/2}\), Speed of neutrons in relativistic mechanics approach

\[ v_{\text{rel}} = 4.4177 \]

\[ v_{\text{error}} := \left( \frac{v_{\text{classical}}}{v_{\text{rel}}} - 1 \right) \cdot 100 \]  

\%error in using classical mechanics approach to calculate neutron velocity

\[ v_{\text{error}} = 0.7965 \% \]

This concludes the problem solution for Homework Problem 2.3!
Problem: What speed (m/s) and kinetic energy (MeV) would a neutron have if its relativistic mass were 10% greater than its rest mass?

Solution: We will use some of the results from Problem 2.3 to calculate the relativistic speed and the kinetic energy of the neutron along with Eq. (2.10).

\[ m_n := 1.0086649233 \text{ amu, Rest mass of neutron} \]

\[ m_{rel} := 1.10 \cdot m_n \text{ amu, Relativistic mass of neutron} \]

\[ c := \sqrt{931.49} \text{ (MeV/amu)}^{1/2}, \text{ Energy-to-mass conversion factor for the speed of light} \]

\textit{Relativistic Mechanics}

\[ T := m_{rel} \cdot c^2 - m_n \cdot c^2 \text{ MeV, Kinetic energy of neutron} \]

\[ T = 93.9561 \]

\[ T = m_{rel} \cdot c^2 - m_n \cdot c^2 = m_n \cdot c^2 \left( \frac{1}{\sqrt{1 - \frac{v_{rel}^2}{c^2}}} - 1 \right) \]

\[ \frac{T}{(m_n \cdot c^2)} + 1 = \sqrt{\frac{1}{1 - \frac{v_{rel}^2}{c^2}}} \]

\[ v_{rel} := \left( 2 \cdot T \cdot m_n \cdot c^2 + T^2 \right)^{1/2} \cdot \frac{c}{T + m_n \cdot c^2} \text{ (MeV/amu)}^{1/2}, \text{ Speed of neutrons in relativistic mechanics approach (see results from Problem 2.3)} \]

\[ v_{rel} = 12.7147 \]
\[ V_{\text{rel}} := v_{\text{rel}} \cdot \sqrt{\frac{(2.998 \times 10^8)^2}{931.49}} \quad \text{m/s, Speed of neutron} \]

\[ V_{\text{rel}} = 1.2490 \times 10^8 \]

It is interesting to note the conversion factor to get from \((\text{MeV/amu})^{1/2}\) to meters/second. Remember that the equivalent conversion for the square of the speed of light, \(c^2\), is \(931.49\) MeV/amu. Obviously, then to get the units for \(c\), the square root of \(931.49\) MeV/amu must be taken and remember that \(c = 2.998 \times 10^8\) meters/second! Always perform a units check to ensure your conversions are executed properly.

This concludes the problem solution for Homework Problem 2.4!
Problem: In a Relativistic Heavy Ion Collider, nuclei of gold are accelerated to speeds of 99.95% the speed of light. These nuclei are almost spherical when at rest; however, as they move past the experimenters they appear considerably flattened in the direction of motion because of relativistic effects. Calculate the apparent diameter of such a gold nucleus in its direction of motion relative to that perpendicular to the motion.

Solution: First we will write down what we know from the problem definition such as $v/c$ and the diameter of a gold nucleus. We will then look at the change in length due to relativistic effects as it may apply to the diameter of the gold nucleus.

$c := 2.998 \times 10^8$ m/s, Speed of light

$\text{fac} := 0.9995$ dimensionless, Ratio of speed of gold nuclei to the speed of light

$v := \text{fac} \cdot c$ m/s, speed of gold nuclei

$A_{Au} := 196.96655$ amu, Atomic weight of gold

$R_0 := 1.25 \times 10^{-13}$ cm, Spherical radius constant (see Eq. (1.7))

$R_{Au} := \frac{R_0 \cdot A_{Au}}{3}$ cm, Calculated radius of gold nucleus per Eq. (1.7) of text

$D_0 := 2 \cdot R_{Au}$ cm, Nucleus diameter as it appears when "stationary"$

D_{exp} := D_0 \sqrt{1 - \text{fac}^2}$ cm, Length (or diameter) as it appears to experimenters watching the gold nuclei fly by

$D_{exp} = 4.5992 \times 10^{-14}$

This concludes the problem solution for **Homework Problem 2.5**!
Problem: Muons are subatomic particles that have the negative charge of an electron but are 206.77 times more massive. They are produced high in the atmosphere by cosmic rays colliding with nuclei of oxygen or nitrogen, and muons are the dominant cosmic-ray contribution to background radiation at the earth's surface. A muon, however, rapidly decays into an energetic electron, existing from its point of view, for only 2.20 $\mu$s, on the average. Cosmic-ray generated muons typically have speeds of about 0.998$c$ and thus should travel only a few hundred meters in air before decaying. Yet muons travel through several kilometers of air to reach the earth's surface. Using the results of special relativity, explain how this is possible. HINT: Consider the atmospheric travel distance as it appears to a muon, and the muon lifetime as it appears to an observer on the earth's surface.

Solution: The lifetime of a muon as measured by an observer on earth is longer due to the time dilation that occurs with relativistic effects. In its own stationary reference frame, the muon has an actual life span of $t_o = 2.20 \mu s$. However, when relativity is considered, the life span of a muon recorded by an observer on earth is determined by Eq. (2.7):

$$t = \frac{t_o}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

For a muon traveling at $0.998c$, the length of time that a muon exists appears to be:

$$t_o := 2.20 \mu s; \text{ actual life span of a muon}$$

$$v_{ratio} := 0.998 \quad \text{v/c; ratio of the muon's speed to the speed of light}$$

$$t := \frac{t_o}{\sqrt{1 - v_{ratio}^2}} \quad \mu s; \text{ life span of a muon as seen on earth}$$

$t = 34.80246$

So, according to an observer on earth, a muon lives for $34.80246 \mu s$. The distance that it can travel at a speed of $0.998c$ before it decays is therefore:
\[ v := 0.998 \cdot 2.998 \cdot 10^8 \quad \text{m/s; speed of a muon with respect to earth} \]

\[ t_{SI} := t \cdot 10^{-6} \quad \text{s; life span of muon with respect to earth} \]

\[ \text{distance} := v \cdot t_{SI} \quad \text{m; distance a muon travels with respect to earth} \]

\[ \text{distance} = 1.041291 \times 10^4 \]

So, considering relativistic effects, the muon can travel about 10 km before decaying when observed by a person on earth. This makes it much more probable for a muon to reach earth's surface before it decays. Now, according to the muon, the earth is traveling toward it at a speed of 0.998c. Therefore, the length between the muon and the earth is contracted by relativistic effects. Suppose the muon was at 10km from earth's surface. Then, the distance it measures to get to the earth is:

\[ L_0 := 10 \cdot 10^3 \quad \text{m; actual distance to the earth's surface} \]

\[ L_{\text{muon}} := L_0 \sqrt{1 - \frac{v \text{ratio}^2}{c^2}} \quad \text{m; apparent distance to the earth's surface with respect to the muon} \]

\[ L_{\text{muon}} = 632.139225 \]

If the muon lives for 2.20 µs, then it should only be able to travel a distance of about 660 m before decaying. If it had to travel 10 km, then it is unlikely to reach the earth's surface before decaying. However, because of relativistic effects it only has to travel about 630 m, which is much more probable before it decays. Therefore, a muon with a lifespan of 2.20 µs is able to reach earth's surface before it decays because it lives longer in a reference frame on earth, and in its own reference frame the distance is shorter!

This concludes the problem solution for Homework Problem 2.6!

Solution: We will use Eq. (2.26) to solve for the scattering angle as follows:

\[ E := 1.0 \quad \text{MeV; Incident energy of gamma ray} \]

\[ E' := E - 0.2 \quad \text{MeV; Scattered gamma ray energy} \]

\[ E_{\text{rest}} := 0.51099890 \quad \text{MeV; Rest energy of an electron (Table 1.5)} \]

\[ \theta_s := \arccos \left[ 1 - E_{\text{rest}} \left( \frac{1}{E'} - \frac{1}{E} \right) \right] \quad \text{radians; Scattering angle of gamma ray as a result of Compton scattering} \]

\[ \theta_s = 0.5110 \]

This concludes the problem solution for Homework Problem 2.7!
Problem: At what energy (in MeV) can a photon lose at most one-half of its energy in Compton scattering?

Solution: We will use Eq. (2.26) to solve this Compton scattering problem as follows:

\[ E \text{ (in MeV)} \] is the incident energy of the photon.

\[ E' = 0.5 \cdot E \] MeV; Energy of scattered photon

\[
\frac{1}{E} - \frac{1}{E' - E} = \left[ \frac{1}{m_e c^2} \cdot (1 - \cos(\theta_s)) \right]^{-1} \text{ (MeV)}^{-1}, \text{ Eq. (2.26)}
\]

Now to find an extrema, i.e., a maxima or minima for the equation above, we will use a first derivative test wrt \( \theta_s \), as \( \theta_s \) is the ONLY variable. We then find that:

\[ E_{\text{rest}} := 0.51099890 \text{ MeV; Rest energy of an electron (Table 1.5)} \]

\[ c := \sqrt{931.49} \text{ (MeV/amu)}^{1/2}, \text{ Speed of light conversion} \]

\[ m_e := \frac{E_{\text{rest}}}{c^2} \text{ amu, Rest mass of an electron} \]

\[
\frac{1}{m_e c^2} \cdot (1 - \cos(\theta_s))
\]

by differentiation, yields
and when we set this equal to zero, we may find that for $\theta_s$ equal to 0 or $\pi$ radians, the function will be an extrema. Testing Eq. (2.26), we find that when $\theta_s$ is equal to $\pi$ radians, we have a maxima, so then we are able to complete the stated problem as follows:

$$\theta_s := \pi$$ radians, Scattering angle that maximizes the reciprocal of the energy difference given in Eq. (2.26)

$$E := \left[ \frac{1}{m_e c^2} \cdot (1 - \cos(\theta_s)) \right]^{-1}$$ MeV, Incident energy at which half the energy is lost in Compton scattering

$$E = 0.25550$$

This concludes the problem solution for Homework Problem 2.8!
Problem: A 1 MeV photon is Compton scattered at an angle of 55°. Calculate

(a) the energy of the scattered photon
(b) the change in wavelength
(c) the recoil energy of the electron

Solution: We will use Eq. (2.26) as follows:

\[ E_1 = 1 \text{ MeV}, \text{ Energy of incident photon} \]

\[ c := \sqrt{931.49} \text{ (MeV/amu)}^{1/2}, \text{ Speed of light conversion} \]

\[ E_{\text{rest}} := 0.51099890 \text{ MeV}; \text{ Rest energy of an electron (Table 1.5)} \]

\[ m_e := \frac{E_{\text{rest}}}{c^2} \text{ amu, Rest mass of an electron} \]

\[ \theta_s := 55 \cdot \frac{\pi}{180} \text{ radians, Scatter angle of photon} \]

\[ E' := \left[ \frac{1}{m_e c^2} \cdot \left(1 - \cos(\theta_s)\right) + \frac{1}{E}\right]^{-1} \text{ (MeV)}^{-1}, \text{ Energy of scattered photon defined in Eq. (2.26)} \]

\[ E' = 0.5451 \]

\[ \lambda_c := 2.4263 \times 10^{-12} \text{ m, Electron Compton wavelength which is equal to } h/m_e c \text{ [note: value after Eq. (2.25) in text is incorrect]} \]

\[ \Delta\lambda := \lambda_c \cdot (1 - \cos(\theta_s)) \text{ m, Change in wavelength of incident photon to when it becomes scattered photon} \]

\[ \Delta\lambda = 1.0346 \times 10^{-12} \]
Now when we look at the Figure 2.5 in the text, we see that momentum must be conserved in the collision between the photon and the electron, and we know that the respective energies can be obtained by multiplying the momentum times $c$, the speed of light. Therefore, the recoil energy of the electron is simply the vector difference of the incident photon energy and the recoil photon energy, $E - E'$ as calculated below:

$$E_{\text{electron\_recoil}} := E - E'$$

MeV, Energy of recoil electron

$E_{\text{electron\_recoil}} = 0.4549$

This concludes the problem solution for Homework Problem 2.9!
Problem: Show the de Broglie wavelength of a particle with kinetic energy \( T \) can be written as:

\[
\lambda = \frac{h}{\sqrt{m_0 \sqrt{T}}} \left( 1 + \frac{m}{m_0} \right)^{-\frac{1}{2}}
\]

where \( m_0 \) is the particle's rest mass and \( m \) is its relativistic mass.

Solution: Quite simply, we will start with Eq. (2.30) in the text, noticing that the constant \( c \), the speed of light, has been eliminated from the equation above and then write an expression for \( c \) based upon Eq. (2.10) in terms of \( T \), \( m_0 \), and \( m \) as follows:

\[
c = \frac{T}{\sqrt{(m-m_0)}}
\]

Speed of light rewritten from Eq. (2.10) of text

\[
\lambda = \frac{h}{p} = \frac{h \cdot c}{\sqrt{T^2 + 2 \cdot T \cdot m_0 \cdot c^2}}
\]

de Broglie wavelength given by Eq. (2.30)

by substitution for \( c \), yields

\[
\lambda = h \cdot \frac{1}{\sqrt{\frac{T}{m-m_0}}} \left( \frac{T}{m-m_0} \right)^{-\frac{1}{2}} \left( \frac{T^2 + 2 \cdot T^2 \cdot \frac{m_0}{m-m_0}}{m-m_0} \right)^{-\frac{1}{2}}
\]

This concludes the problem solution for Homework Problem 2.10!

[HW_2_10.xmcd]