Since December 2, 1942, when the first human-made nuclear reactor produced a self-sustaining chain reaction, several hundred different types of reactor systems have been constructed. Despite the many possible differences in design, there are a number of general features which all reactors have in common. The heart of every reactor is an active core in which the fission chain reaction is sustained. The active core contains (1) fissile fuel which through its fissioning is the main source of neutrons, (2) moderator material if the fission neutrons are to slow down, (3) coolant if the heat generated by the fissions is to be removed from the core, and (4) structural material which maintains the physical integrity of the core. Surrounding the active core is usually either a reflector whose purpose is to scatter neutrons back towards the core or a blanket region which captures neutrons leaking from the core to produce useful isotopes such as $^{60}$Co or $^{239}$Pu. The reactor core and reflector/blanket are in turn surrounded by a shield to minimize radiation reaching personnel and equipment near the reactor. Finally, all reactors must have a means of control to allow the chain reaction to be started up, maintained at some desired level, and safely shutdown.

Reactors are broadly classified according to the energy of the neutrons that cause most of the fissions. In a fast reactor, the fast fission neutrons do not slow down very much before they are absorbed by the fuel and cause the production of a new generation of fission neutrons. By contrast, in a thermal reactor almost all fissions are caused by neutrons that have slowed down and are moving with speeds comparable to those of the atoms of the core material, i.e., the neutrons are in thermal equilibrium with the surrounding material.

In this chapter, the basic principles of nuclear reactors and fission chain reactions are discussed. Initially, we consider steady-state neutron populations in a reactor core, and seek methods to quantify the conditions necessary for a self-sustaining chain reaction with a constant neutron population and fission power release. In particular, we concentrate on “thermal” reactors, although the principles for fast reactors are quite similar. Later in the chapter, we consider the dynamics of reactors as the power increases or decreases in response to physical changes in the reactor as a result of externally applied changes or from feedback effects.
10.1 Neutron Moderation

In a thermal reactor the fast fission neutrons lose their kinetic energy primarily through elastic scattering from moderator nuclei with small mass numbers. In our earlier discussion of the kinematics of neutron elastic scattering from a stationary nucleus,\(^1\) we found that the energy of the scattered neutron is between \(E_{\text{max}} = E\) and \(E_{\text{min}} = \alpha E\) where \(E\) is the energy of the incident neutron and \(\alpha \equiv (A - 1)^2/(A + 1)^2\). The number of scatters, on the average, required to bring a neutron with initial energy \(E_1\) to a lower energy \(E_2\) is given by Eq. (6.30). A summary of these important properties for several moderators is given in Table 6.1.

From the values in Table 6.1, it is seen that scattering nuclei with small \(A\) numbers cause greater average energy loss and thus thermalize fast neutrons with fewer scatters than do nuclei with large mass numbers. In addition to having a small \(A\) number, a good moderator should have a large scattering cross section \(\Sigma_s\) (to encourage scattering) and a small absorption cross section \(\Sigma_a\) (to avoid loss of neutrons before they can cause fissions). In all thermal reactors, a large portion of the core material is the moderator, usually light or heavy water, graphite, or beryllium. In addition to acting as a moderator, light water can also serve as the main coolant in a power reactor, and, for this reason, it is not surprising that light water reactors (LWRs) are the dominant type of power reactors in service today.

10.2 Thermal-Neutron Properties of Fuels

In thermal reactors, only fissile isotopes such as \(^{233}\text{U}\), \(^{235}\text{U}\) and \(^{239}\text{Pu}\) can be used. By far the most widely used nuclear fuel is uranium dioxide, with the uranium enriched in \(^{235}\text{U}\) from its natural 0.720 atom-% to several percent. Only some heavy-water and graphite-moderated reactors can use natural uranium as fuel. Most reactors use uranium that has been enriched, typically 2 to 3\%, in \(^{235}\text{U}\). The fissile nuclide \(^{239}\text{Pu}\) is created during the operation of a nuclear power reactor whose fuel contains \(^{238}\text{U}\) (see Section 6.5.3), and at the end of fuel life (typically three years), almost half of the power is generated by the fission of \(^{239}\text{Pu}\). In some power reactors, \(^{239}\text{Pu}\) is mixed with enriched uranium in the form of a "mixed oxide" fuel. Some important properties of nuclear fuels are presented in Table 10.1. The data lead to the following observations:

1. \(^{233}\text{U}\) has the largest value of \(\eta\), the number of fission neutrons produced per thermal neutron absorbed, and hence is the best prospect for a thermal breeder reactor, one in which more fissile fuel is produced by neutron absorptions than is consumed in the chain reaction. A breeder reactor needs an \(\eta\) of at least two since one neutron is needed to sustain the chain reaction and one neutron must be absorbed in the fertile material to breed a new fissile fuel atom. Fertile materials are those such as \(^{232}\text{Th}\) and \(^{238}\text{U}\) that, upon thermal neutron absorption, may yield fissile materials (see Section 6.5.3).

2. Although the plutonium isotopes produce almost 3 fission neutrons per thermal fission, their relatively high radiative capture (\(n,\gamma\)) cross sections (indicated by the relatively large \(\sigma_r/\sigma_f\) ratio) result in low values of \(\eta\). However,\(^{1}\) The thermal motion of the nucleus is negligible compared to the speed of fast neutrons.
Table 10.1. Thermal-neutron properties of important fuel isotopes. Cross sections are at 0.0253 eV (2200 m/s).

<table>
<thead>
<tr>
<th>Nuclide</th>
<th>$\sigma_a$ (b)</th>
<th>$\sigma_\gamma$</th>
<th>$\sigma_f$</th>
<th>$\frac{\sigma_a}{\sigma_f}$</th>
<th>$\nu$</th>
<th>$\eta = \nu \frac{\sigma_f}{\sigma_a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{232}$Th</td>
<td>5.13</td>
<td>5.13</td>
<td>--</td>
<td>1.0</td>
<td>0.0</td>
<td>--</td>
</tr>
<tr>
<td>$^{233}$U</td>
<td>575</td>
<td>46</td>
<td>529</td>
<td>0.087</td>
<td>2.49</td>
<td>2.29</td>
</tr>
<tr>
<td>$^{235}$U</td>
<td>687</td>
<td>99.3</td>
<td>587</td>
<td>0.169</td>
<td>2.42</td>
<td>2.07</td>
</tr>
<tr>
<td>$^{238}$U</td>
<td>2.73</td>
<td>2.73</td>
<td>--</td>
<td>1.0</td>
<td>0.0</td>
<td>--</td>
</tr>
<tr>
<td>$^{239}$Pu</td>
<td>1020</td>
<td>271</td>
<td>749</td>
<td>0.362</td>
<td>2.87</td>
<td>2.11</td>
</tr>
<tr>
<td>$^{240}$Pu</td>
<td>289.5</td>
<td>289.5</td>
<td>0.064</td>
<td>--</td>
<td>0.0</td>
<td>--</td>
</tr>
<tr>
<td>$^{241}$Pu</td>
<td>1378</td>
<td>363</td>
<td>1015</td>
<td>0.358</td>
<td>2.92</td>
<td>2.15</td>
</tr>
</tbody>
</table>

for fissions caused by neutrons with energies above several hundred keV, the $\eta$ for both $^{239}$Pu and $^{241}$Pu is greater than 3. Thus fast reactors using plutonium as fuel are attractive as breeder reactors.

3. The fertile isotopes $^{232}$Th and $^{238}$U have absorption cross sections of about 1% or less than those of their conversion fissile isotopes $^{233}$U and $^{239}$Pu.

4. The fertile isotope $^{240}$Pu has a large capture cross section for the production of the fissile isotope $^{241}$Pu.

5. Although not shown in this table, the fission and absorption cross sections for reactions of high energy fission neutrons with fissile isotopes are several hundred times less than for reactions with thermal neutrons (see, for example, Fig. 7.10).

10.3 The Neutron Life Cycle in a Thermal Reactor

In a thermal reactor, some of the fast neutrons ($\sim 2$ MeV) born from fission slow down to thermal energies ($\sim 0.025$ eV), are absorbed by the fuel, cause the fuel to fission, and thus produce a second generation of fast neutrons. The remainder of the fast neutrons suffer a variety of fates that do not result in fission.

Consider the life cycle shown below for one generation of neutrons in a thermal reactor. Here $n$ fast neutrons are introduced into the core. These fast neutrons can cause a few fast fission events (thereby creating some second generation neutrons), can leak from the core, or can be absorbed while slowing down to thermal energies. Those neutrons that reach thermal energies begin to diffuse throughout the core. Some of these thermal neutrons may be lost by leaking from the core or by being absorbed by the “nonfuel.” Most thermal neutrons, however, are absorbed in the fuel, but only a certain fraction of these absorptions cause the fuel to fission thereby releasing $\nu$ fast second-generation neutrons per fission. Thus, at the end of the cycle there is a new generation of $n'$ fast neutrons which begin the cycle again. Clearly for the neutron population to remain constant cycle after cycle, $n'$ must equal $n$, i.e., the chain reaction must be self-sustaining.
10.3.1 Quantification of the Neutron Cycle

To quantify the neutron life cycle in a thermal reactor, as shown in Fig. 10.1, we define the following six factors.

1. fast fission factor \( \epsilon \): To account for fast fission, we define \( \epsilon \) as the ratio of the total number of fast neutrons produced by both thermal and fast fission to the number produced by thermal fission alone. This quantity is quite difficult to calculate accurately. It is generally quite small, varying typically between 1.02 and 1.08 for reactors fueled with natural or slightly enriched uranium.

Fast fission is almost completely due to \( ^{238}\text{U} \) which has a fission threshold of about 1 MeV (see [Fig. 7.1]). Thus in a core using highly enriched uranium, \( \epsilon \) is very close to unity. Also, in a very dilute homogeneous mixture of uranium in a moderator, \( \epsilon \) is very close to unity since a fast fission neutron is far more likely to first encounter a moderator atom and lose some energy than it is to encounter an atom of \( ^{238}\text{U} \).

The fast fission factor is generally higher in a heterogeneous core, in which the fuel is lumped together, compared to a homogeneous core. This increase for a heterogeneous core is a result of the fast fission neutrons having a better chance of encountering \( ^{238}\text{U} \) atoms before they begin slowing down and hence a better chance to cause fast fission. Typically, \( \epsilon \) in a heterogeneous core is 5–10% higher than in an equivalent homogeneous mixture.
2. Resonance Escape Probability $p$: To account for fast neutron absorption we define $p$ as the probability that a fast fission neutron slows to thermal energies without being absorbed. In uranium fueled reactors, most neutrons that are absorbed while slowing down are absorbed by $^{238}\text{U}$ which has very large absorption resonances in its cross section in the slowing down energy region. Calculation of $p$ for a core is also a very difficult task because of the complicated variation of core-material cross section with neutron energy in the slowing down energy region. Many empirical results have been obtained and are often used in preliminary analyses (see Lamarsh [1966] and Lamarsh and Baratta [2001]).

For reactors fueled with fully enriched $^{235}\text{U}$ or $^{239}\text{Pu}$, $p$ is very close to unity. For homogeneous cores with uranium of only a few percent enrichment, $p$ can vary between 0.9 and 0.6 depending on the fuel-to-moderator ratio. In heterogeneous cores, $p$ is considerably higher than in a homogeneous core of the same fuel-to-moderator ratio, typically varying between 0.8 and 0.9. By lumping the fuel atoms together, the neutrons can slow down in the moderator away from the fuel and, consequently, are less likely to encounter a $^{238}\text{U}$ atom and be absorbed. By contrast, in a homogeneous core material, as neutrons slow past the $^{238}\text{U}$ resonance energies they are more likely to be near a $^{238}\text{U}$ atom and thus to be absorbed.

3. Thermal Utilization $f$: Because not all thermal neutrons are absorbed by the fuel, we define $f$ as the probability that, when a thermal neutron is absorbed, it is absorbed by the “fuel” ($F$) and not by the “nonfuel” ($NF$). Equivalently, $f$ is the ratio of the average thermal neutron absorption rate in the fuel to the total thermal neutron absorption rate in the fuel and nonfuel. Mathematically,

$$ f = \frac{\sum_a F \phi^F V^F}{\sum_a F \phi^F V^F + \sum_a NF \phi^{NF} V^{NF}} = \frac{\sum_a F}{\sum_a F + \sum_a NF (V^{NF}/V^F) (\phi^{NF}/\phi^F)}, $$

where $\phi^F$ and $\phi^{NF}$ are the average thermal flux densities in the fuel (of volume $V^F$) and nonfuel (with volume $V^{NF}$), respectively. For a homogeneous core $\phi^F = \phi^{NF}$ and $V^F = V^{NF}$ so that

$$ f = \frac{\sum_a F}{\sum_a F + \sum_a NF} = \frac{\sigma_a^F}{\sigma_a^F + \sigma_a^{NF} (N^{NF}/N^F)}, $$

where $N^{NF}/N^F$ is the ratio of fuel-to-nonfuel atomic concentrations. The value of $f$ can range from near zero for a very dilute fuel mixture to unity for a core composed only of fuel.

4. Thermal Fission Factor $\eta$: We define $\eta$ as the number of fast fission neutrons produced per thermal neutron absorbed by the “fuel.” Equivalently, $\eta$ is the average number of neutrons per thermal fission ($\nu$) times the probability a fission occurs when a thermal neutron is absorbed by the fuel, i.e.,

$$ \eta = \nu \frac{\sum_a F}{\sum_a F}. $$
Since non-fuel has a zero fission cross section, \( \Sigma_f = \Sigma_f \), the total fission cross section of the core material. Notice this factor is a property of the fuel material alone and is unaffected by the type and amount of nonfuel material in the core.

This factor must be greater than unity if a self-sustaining chain reaction is to be realized. Values of \( \eta \) for isotopically pure fuels are given in Table 10.1. For fuel composed of a mixture of isotopes we need to calculate the macroscopic cross sections for each isotope and add the results to get the total cross section (see Example 10.2).

**Example 10.1:** What is the thermal utilization factor in a mixture of graphite and natural uranium with a carbon-to-uranium atom-ratio \( \frac{N_C}{N_U} \) of 450?

Natural uranium contains the three isotopes \(^{234}\text{U}\), \(^{235}\text{U}\) and \(^{238}\text{U}\) with atomic abundances (see Appendix A.4) \(f_{^{234}} = 0.0055\%\), \(f_{^{235}} = 0.720\%\), \(f_{^{238}} = 99.2745\%\), respectively. From Appendix C.2 \(\sigma_a^{^{234}} = 103.4\) b, \(\sigma_a^{^{235}} = 687\) b, and \(\sigma_a^{^{238}} = 2.73\) b. Thus the total thermal macroscopic cross section for the uranium in the carbon-uranium mixture is

\[
\Sigma_U = f_{^{234}}\sigma_a^{^{234}} + f_{^{235}}\sigma_a^{^{235}} + f_{^{238}}\sigma_a^{^{238}}\frac{N_U}{N} = 7.662 N_U \text{ cm}^{-1}.
\]

We can assume that graphite is \(^{12}\text{C}\) since other naturally occurring carbon isotopes have very low abundances and cross sections. From Table C.2 \(\sigma_a^{^{12}} = 0.0034\) b so that

\[
\Sigma_C = N_C \sigma_a^{^{12}} = 0.0034 N_C.
\]

Then from Eq. (10.2)

\[
f = \frac{\Sigma_T}{\Sigma_U + \Sigma_C} = \frac{7.662 N_U}{7.662 N_U + 0.0034 N_C} = \frac{7.662}{7.662 + 0.0034(150/450)} = 0.8336.
\]

**Example 10.2:** What is the thermal fission factor \( \eta \) for uranium enriched to 2 atom-% in \(^{235}\text{U}\), the remainder being \(^{238}\text{U}\)? From Eq. (10.3) and the data in Table 10.1 we have

\[
\eta = \frac{\nu^{^{235}} \Sigma_f^{^{235}}}{\Sigma_f} = \frac{\nu^{^{235}} N^{^{235}} \sigma_f^{^{235}}}{N^{^{234}} \sigma_a^{^{234}} + N^{^{238}} \sigma_a^{^{238}}} = \frac{\nu^{^{235}} \sigma_f^{^{235}}}{\sigma_a^{^{235}} + \sigma_a^{^{238}}(N^{^{238}}/N^{^{235}})}
\]

\[
= \frac{2.42 \times 587}{687 + 2.73(98/2)} = 1.73.
\]

**5. thermal non-leakage probability \( P_{NL}^{th} \):** To account for thermal neutrons leaking from the core, we define \( P_{NL}^{th} \) as the probability a thermal neutron does not leak from the core before it is absorbed. It can be shown [Lamarsh and Baratta 2001] that \( P_{NL}^{th} \) can be estimated for a bare core from

\[
P_{NL}^{th} = \frac{1}{1 + L^2 B^2}, \quad (10.4)
\]
Table 10.2. Moderator properties for thermal (0.00253 eV) neutrons. \( L \) is the thermal diffusion length and \( \tau \) is the Fermi age from fission to thermal energies. Source: [ANL 1963].

<table>
<thead>
<tr>
<th>Moderator</th>
<th>Density ( \rho ) (g cm(^{-3}))</th>
<th>( \Sigma_a ) (cm(^{-1}))</th>
<th>( D ) (cm)</th>
<th>( L^2 ) (cm(^2))</th>
<th>( L ) (cm)</th>
<th>( \tau ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H(_2)O</td>
<td>1.00</td>
<td>0.0197</td>
<td>0.16</td>
<td>8.1</td>
<td>2.85</td>
<td>27</td>
</tr>
<tr>
<td>D(_2)O</td>
<td>1.10</td>
<td>2.9 \times 10^{-5}</td>
<td>0.87</td>
<td>30,000</td>
<td>170</td>
<td>131</td>
</tr>
<tr>
<td>Be</td>
<td>1.85</td>
<td>1.04 \times 10^{-3}</td>
<td>0.50</td>
<td>480</td>
<td>21</td>
<td>102</td>
</tr>
<tr>
<td>BeO</td>
<td>2.96</td>
<td>6.0 \times 10^{-4}</td>
<td>0.47</td>
<td>790</td>
<td>28</td>
<td>100</td>
</tr>
<tr>
<td>C</td>
<td>1.60</td>
<td>2.4 \times 10^{-4}</td>
<td>0.84</td>
<td>3500</td>
<td>59</td>
<td>368</td>
</tr>
</tbody>
</table>

where \( L \) is the thermal diffusion length and \( B_c^2 \) is the critical buckling. The evaluation of these two parameters is discussed next.

The square of the thermal diffusion length \( L^2 \equiv D/\Sigma_a \) where \( D \) is the thermal diffusion coefficient and \( \Sigma_a \) is the thermal macroscopic absorption cross section of the core material. It can be shown that \( L \) is one-half of the average distance a thermal neutron diffuses from the point where it becomes thermal to the point at which it is absorbed in an infinite medium of core material. Values of \( D, \Sigma_a, \) and \( L^2 \) are given in Table 10.2 for several pure moderators. When material such as fissile fuel is added to a moderator to make core material, the value of \( L^2 \) decreases significantly while \( D \) remains relatively constant for dilute fuel-moderator mixtures. Thus, for a homogeneous mixture of fuel \( (F) \) and moderator \( (M) \)

\[
L^2 = \frac{D}{\Sigma_a} = \frac{D^M}{\Sigma_a^F + \Sigma_a^M} = \frac{D^M}{\Sigma_a^F} \frac{\Sigma_a^F}{\Sigma_a^F + \Sigma_a^M} = L^2_M \left(1 - \frac{\Sigma_a^F}{\Sigma_a^F + \Sigma_a^M}\right) = L^2_M (1 - f).
\]

(10.5)

The critical buckling, \( B_c^2 \), discussed in Addendum 1, depends only on the geometry and size of the reactor core. Expressions for \( B_c^2 \) are given in Table 10.6 as \( B_g^2 \) for some simple core geometries. For example, a spherical core of radius \( R \) has a critical buckling of \( B_c^2 = (\pi/R)^2 \). Thus as the core increases in size, \( B_c^2 \rightarrow 0 \) and \( P_{NL}^F \rightarrow 1 \), i.e., there is no leakage.

6. fast non-leakage probability \( P_{NL}^F \): The leakage of fast neutrons from the core is accounted for by \( P_{NL}^F \), defined as the probability a fast neutron does not leak from the core as it slows to thermal energies. This factor can be estimated for a bare core from [Lamarsh and Baratta 2001]

\[
P_{NL}^F = e^{-B_c^2 \tau},
\]

(10.6)

where \( \tau \) is the Fermi age to thermal energies and can be interpreted as one-sixth the mean squared distance between the point at which a fast fission neutron is born and begins to slow down and the point at which it reaches thermal energies. In Table 10.2 values of \( \tau \) are given for pure moderators. In dilute fuel-moderator mixtures, \( \tau \) changes little from its value for the pure moderator. Again we see from Eq. (10.6) that, as the core size increases, \( B_c^2 \rightarrow 0 \), and the non-leakage probability \( P_{NL}^F \rightarrow 1 \).
With the above six definitions, the number of neutrons at various stages in the life cycle can be calculated as shown in Fig. 10.2.

**Example 10.3:** What are the thermal and fast neutron non-leakage probabilities for a bare spherical reactor core of radius \( R = 120 \) cm composed of a homogeneous mixture of graphite and \(^{235}\)U in an atomic ratio of 40,000 to 1? The geometric buckling for the spherical core is \( B_{g}^{2} = \left(\frac{\pi}{R}\right)^{2} = \left(\frac{\pi}{120}\right)^{2} = 6.85 \times 10^{-4} \text{ cm}^{-2} \). For such a dilute mixture of uranium in graphite, the Fermi age to thermal energies is the same as that in a pure graphite medium, namely \( \tau = 368 \text{ cm}^{2} \). Thus, from Eq. (10.6), the fast-neutron nonleakage probability is

\[
R_{NL}^{f} = \exp(-B_{g}^{2}\tau) = \exp(-6.85 \times 10^{-4} \times 368) = 0.777.
\]

From Example 10.4, the thermal utilization for this core mixture is \( f = 0.835 \), so that the square of the thermal diffusion length \( L^{2} = L_{C}^{2}(1-f) = 3500(1-0.835) = 578 \). The thermal-neutron nonleakage probability, given by Eq. (10.2), is thus

\[
R_{NL}^{th} = \frac{1}{1 + L^{2}B_{g}^{2}} = \frac{1}{1 + (578)(6.85 \times 10^{-4})} = 0.716.
\]

---

**Figure 10.2.** The neutron life cycle in a thermal reactor showing the calculation of the major contributions to the loss and gain of second-generation neutrons.

### 10.3.2 Effective Multiplication Factor

From Fig. 10.2 the neutron gain or *effective multiplication factor* per neutron cycle is seen to be

\[
k_{eff} = \frac{\text{no. of neutrons at some point in the cycle}}{\text{no. of neutrons at the same point in the previous cycle}}
\]
or

\[
\frac{n'}{n} = \frac{n\eta f P_{\text{NL}}^L P_{\text{NL}}^R}{n}.
\]

or

\[
k_{\text{eff}} = \eta \epsilon P_{\text{NL}}^L P_{\text{NL}}^R.
\] (10.7)

Clearly, if \( k_{\text{eff}} < 1 \), the initial neutron population decreases with each cycle until no neutrons are left. Such a system is said to be subcritical, requiring an independent source of neutrons to maintain a steady neutron population. However, if \( k_{\text{eff}} > 1 \), the number of neutrons continually increases with each cycle and the system is said to be supercritical. For the special case that \( k_{\text{eff}} = 1 \), the number of neutrons remains constant cycle after cycle and the system is said to be self-sustaining or critical.

For an infinite medium, there is no neutron leakage and \( P^L_{\text{NL}} \) and \( P^R_{\text{NL}} \) are unity. Thus the multiplication factor is the infinite medium multiplication factor

\[
k_{\infty} = \eta \epsilon f.
\] (10.8)

This result is known as the “four-factor formula” of reactor physics. Notice that \( k_{\infty} \) is a property of only the core material and is independent of the size and shape of the core.

These formulas for \( k_{\infty} \) and \( k_{\text{eff}} \) allow us to calculate the criticality state of an assembly. More important, they provide a conceptual framework for assessing the criticality effect of some proposed change to an assembly. For example, inserting a thermal-neutron absorbing material such as a control rod into a critical core primarily affects \( f \) by making it smaller and, consequently, making the core subcritical.

The typical variation of \( k_{\text{eff}} \) and its four factors with the fuel-to-moderator ratio of the core material is shown in Fig. 10.3. Notice that there is a optimal fuel-to-moderator ratio that maximizes the value of \( k_{\text{eff}} \). This maximum value of \( k_{\text{eff}} \) must be greater than unity to compensate for neutron leakage if a core of finite size is to become critical.

---

**Example 10.4:** What is \( k_{\infty} \) of a homogeneous mixture of \(^{235}\text{U}\) and graphite with an atomic uranium to carbon ratio of 1 to 40,000? For such a dilute mixture of fully enriched uranium and carbon, \( \eta \approx 1 \), so that \( k_{\infty} = \eta f \). From [Table 10.1](#) \( \eta = 2.07 \) for pure \(^{235}\text{U}\). With Eq. (10.2) and data from Tables 10.1 and C.1

\[
f = \frac{\sigma_{\text{235}}^U}{\sigma_{\text{235}}^U + \sigma_{\text{C}}^C (N^C/N^{235})} = \frac{687}{687 + 0.0034(40,000)} = 0.835.
\]

Then

\[
k_{\infty} = \eta f = 2.07 \times 0.835 = 1.728.
\]
Figure 10.3. Variation of $k_{\text{eff}}$ and its factors with the fuel-to-moderator ratio. This example is for a homogeneous mixture of water and 2%-enriched uranium. Here $N^F/N^M$ = atom density of uranium to molecular density of water.

Example 10.5: What is the radius $R$ of a critical bare sphere composed of a homogeneous mixture of $^{235}$U and graphite with a uranium to carbon atom ratio of 1 to 40,000? For criticality, we require

$$k_{\text{eff}} = \frac{ep\eta f}{1 + L^2B^2} \exp(-B^2\tau) = 1.$$  

From Example 10.4, $ep = 1$ and $\eta f = 1.73$. From Example 10.3, $L^2 = 578$ cm\(^2\) and $\tau = 368$ cm\(^2\) so that the criticality condition is

$$k_{\text{eff}} = 1 = \frac{1.728}{1 + 578B^2} \exp(-368B^2).$$

Solution for $B^2 \equiv (\pi/R)^2$ by "trial and error" yields $B^2 = 6.358 \times 10^{-4}$ cm\(^{-2}\). From this result we find the critical radius to be $R = 125$ cm.
Example 10.6: What is the mass of $^{235}$U needed for the critical assembly of Example 10.5? Since $N_i = \rho V_n / A_i = (m_i / V_i) N_n / A_i$ and for a homogeneous system $V^{^{235}} = V^C \equiv V$ we have

$$\frac{N^{^{235}}}{N^C} = \frac{m^{^{235}} A^C}{m^C A^{235}} \quad \text{or} \quad m^{^{235}} = m^C N^{^{235}} A^{235} \frac{A^C}{N^C}.$$  

Since we a have a very dilute mixture of $^{235}$U in graphite, $m^C \simeq \rho^C V$ so that for the bare spherical core of Example 10.5 with a radius $R = 125$ cm

$$m^{^{235}} = \left(\frac{4}{3} \pi R^3 \rho^C\right) \frac{N^{^{235}} A^{235}}{N^C A^C} = \left(\frac{4}{3} \pi (125)^3 \times 1.60 \right) \frac{1}{40,000} \frac{235}{12} = 6.41 \text{ kg.}$$

10.4 Homogeneous and Heterogeneous Cores

The least expensive fuel to use in a reactor assembly is natural uranium (0.72 atom-% $^{235}$U). However, $k_\infty$ for a pure natural uranium core would be very small since a fast fission neutron would lose so little energy in each scatter from a uranium nucleus that over 2000 scatters would be required to slow the neutron to thermal energies (see Table 6.1). Such a neutron would thus spend considerable time in the energy regions at which $^{238}$U has large absorption cross sections, and hence the neutron would have little chance of reaching thermal energies. Thus for a pure natural uranium core, the resonance escape probability $p$ would be very small so that $k_\infty$ would be much less than unity.

To use natural uranium as a fuel, it is necessary to slow the fission neutrons more quickly to thermal energies by using a light mass material as a moderator. Since fewer scatters from nuclides with small mass number are needed to slow fission neutrons, it is less likely that neutrons are absorbed while slowing down. Thus $p$ is increased significantly, and $k_\infty$ can approach unity. Conceptually, the simplest assembly is a homogeneous mixture of natural uranium and a moderator material. For such a mixture there is an optimum ratio of moderator-to-uranium atomic concentrations, $(N^M / N^U)_{opt}$, that produces the maximum possible value of $k_\infty$. If $N^M / N^U$ is too small, there is too little moderation and $p$ is very small. On the other hand, if $N^M / N^U$ is too large the thermal neutrons are not absorbed easily by the fuel and $f$ is small. In Table 10.3 the maximum $k_\infty$ achievable with the optimum $N^M / N^U$ ratio is shown. Only for heavy water as the moderator is it possible, in principle, to build a critical homogeneous reactor using natural uranium as fuel.

<table>
<thead>
<tr>
<th>Moderator</th>
<th>$(N^M / N^U)_{opt}$</th>
<th>$\epsilon$</th>
<th>$\eta$</th>
<th>$f$</th>
<th>$p$</th>
<th>$k_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>H$_2$O</td>
<td>1.64</td>
<td>1.057</td>
<td>1.322</td>
<td>0.873</td>
<td>0.723</td>
<td>0.882</td>
</tr>
<tr>
<td>D$_2$O</td>
<td>272</td>
<td>1.000</td>
<td>1.322</td>
<td>0.954</td>
<td>0.914</td>
<td>1.153</td>
</tr>
<tr>
<td>Be</td>
<td>181</td>
<td>1.000</td>
<td>1.322</td>
<td>0.818</td>
<td>0.702</td>
<td>0.759</td>
</tr>
<tr>
<td>C</td>
<td>453</td>
<td>1.000</td>
<td>1.322</td>
<td>0.830</td>
<td>0.718</td>
<td>0.767</td>
</tr>
</tbody>
</table>
However, small critical reactors using all these moderators have been built. The trick for obtaining a system with $k_{eff} > 1$ is (1) to use fuel enriched in $\text{^{235}U}$ and/or (2) construct a heterogeneous core in which the fuel is lumped together and embedded in a matrix of the moderator. The first technique is expensive (even though enrichments of only a few percent are sufficient) because special facilities must be constructed to separate $\text{^{238}U}$ from $\text{^{235}U}$. Sometimes, simply lumping the fuel together is sufficient to create a core with a value of $k_{\infty}$ sufficiently greater than unity.

This increase in $k_{\infty}$ for a heterogeneous core arises primarily as a result of a great increase in the resonance escape probability. By lumping the fuel (and hence the $\text{^{238}U}$ responsible for resonance absorption), the fast neutrons are thermalized in the moderator away from the $\text{^{238}U}$ and hence they can slow through the energy ranges of the $\text{^{238}U}$ resonances with little likelihood of being captured. Only those neutrons reaching the resonance energies near a fuel lump are in danger of being absorbed. The lumping of fuel can increase $\phi$ from about 0.7 for a homogeneous core to values greater than 0.9.

The fast fission factor $\epsilon$ also tends to increase slightly in a heterogeneous core. Fast neutrons born in the fuel lumps have a greater probability of causing fast fissions in $\text{^{238}U}$ if they are surrounded by only uranium atoms. In a homogeneous system, a fast fission neutron may first encounter a moderator atom, scatter, and lose so much energy that it is no longer capable of causing fast fission. However, the increase in $\epsilon$ for a heterogeneous core is small, typically only a few percent.

For a given ratio $N^F/N^\text{NF}$, $f$ in a heterogeneous core is smaller than in a homogeneous core. This can be seen from Eq. (10.1) since in a heterogeneous core, the average thermal flux in the fuel is less than in the nonfuel (moderator) so that $\phi^\text{NF}/\phi^F > 1$. For a homogeneous core $\phi^\text{NF}/\phi^F = 1$ and $f$ is consequently greater.

The one factor in $k_{\infty}$ that decreases in a heterogeneous core is the thermal utilization $f$. From Eq. (10.1)

$$f = \frac{\Sigma_a^F}{\Sigma_a^F + \Sigma_a^{NF}(V^{NF}/V^F)(\phi^{NF}/\phi^F)}$$

$$= \frac{\sigma_a^F}{\sigma_a^F + \sigma_a^{NF}(N^{NF}/N^F)(V^{NF}/V^F)(\phi^{NF}/\phi^F)}.$$  

The value of $(N^{NF}/N^F)(V^{NF}/V^F)$ is the same as in a homogeneous core with the same fuel-to-nonfuel ratio; however, in a heterogeneous core, the thermal flux in the fuel lumps is depressed compared to the moderator (i.e., $\phi^{NF}/\phi^F < 1$), as a consequence of the relatively large absorption rate in the fuel lump compared to the moderator. In essence, the inside of the fuel lump is shielded by the outer layers of the fuel and, hence, does not have as good a chance at absorbing neutrons as would be the case in the homogeneous core.

Finally, the thermal fission factor $\eta$ is a property of the fuel alone and is unaffected by the lumping of fuel. As an example of the importance of lumping the fuel, consider the heterogeneous square lattice shown in Fig. 10.4. The center of each graphite lattice cell, of size $a \times a$, contains a cylindrical fuel rod of natural uranium metal. The fuel rod is infinitely long and has a radius of 1.25 cm. For this graphite-uranium lattice, the fast fission factor does not vary significantly with cell
radius and is constant at $\epsilon = 1.027$. The thermal fission factor is that for natural uranium, $\eta = 1.32$. In Table 10.4, the variation with the cell radius of $k_\infty$ and its factors is shown. Notice that the optimum cell size yields a value for $k_\infty$ greater than unity.

### 10.5 Reflectors

Most reactor cores are surrounded by some material that has a high scattering-to-absorption cross section ratio (typical of moderators). This material, called a reflector, is used for two purposes. First, it reflects some of the neutrons which would escape or leak from a bare core back into the core, thereby increasing the non-leakage probabilities $P^h_i$ and $P^f_i$. This effect is important for small experimental assemblies so as to reduce the amount of fissile fuel needed for criticality.

However, for large power reactors the non-leakage probabilities are very close to unity, and the presence of a reflector has only a very small influence on $k_{\text{eff}}$. More important for power reactors, a reflector tends to raise the thermal flux density

---

**Table 10.4.** Variation of core parameters with cell size for the natural uranium and graphite core shown in Fig. 10.4.

<table>
<thead>
<tr>
<th>Pitch $a$ (cm)</th>
<th>$\epsilon$</th>
<th>$\eta$</th>
<th>$f$</th>
<th>$p$</th>
<th>$k_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>1.027</td>
<td>1.322</td>
<td>0.972</td>
<td>0.742</td>
<td>0.979</td>
</tr>
<tr>
<td>16</td>
<td>1.027</td>
<td>1.322</td>
<td>0.947</td>
<td>0.848</td>
<td>1.090</td>
</tr>
<tr>
<td>20</td>
<td>1.027</td>
<td>1.322</td>
<td>0.916</td>
<td>0.900</td>
<td>1.120</td>
</tr>
<tr>
<td>21</td>
<td>1.027</td>
<td>1.322</td>
<td>0.907</td>
<td>0.909</td>
<td>1.121</td>
</tr>
<tr>
<td>22</td>
<td>1.027</td>
<td>1.322</td>
<td>0.898</td>
<td>0.917</td>
<td>1.119</td>
</tr>
<tr>
<td>26</td>
<td>1.027</td>
<td>1.322</td>
<td>0.860</td>
<td>0.940</td>
<td>1.088</td>
</tr>
<tr>
<td>30</td>
<td>1.027</td>
<td>1.322</td>
<td>0.818</td>
<td>0.955</td>
<td>1.060</td>
</tr>
</tbody>
</table>

---

Figure 10.4. Cross-section of a heterogeneous core. Each unit cell of pitch $a$ contains a 1.25-cm radius fuel rod (black circles) of natural uranium metal. The remainder of each lattice cell is graphite.
(and hence the power density) near the core edges, thereby decreasing the peak-to-average power density in the core (see Fig. 10.5). For heat-transfer purposes it is desirable to maintain as constant a thermal flux profile (and hence a power density profile) across the core as possible. To flatten the power density profile even further, most power reactors place fuel with higher concentrations of $^{235}\text{U}$ near the periphery of the core.

![Figure 10.5. The thermal neutron flux profile in a bare and reflected reactor.](image)

### 10.6 Reactor Kinetics

One of the most important aspects in the design of nuclear reactors is the dynamic response of a reactor to changes in $k_{\text{eff}}$. To vary the power level of a reactor or to shut the reactor down, there must be a mechanism to make $k_{\text{eff}}$ vary about the critical value of unity. For example, the insertion (or withdrawal) of rods composed of material with a large absorption cross section for thermal neutrons (e.g., cadmium or indium) decreases (or increases) the thermal utilization factor $f$, thereby making $k_{\text{eff}}$ decrease (or increase). Such control rods are used in most reactors to control the neutron chain reaction. Similarly, an accident or unforeseen event may cause some property of the core to change which, in turn, alters $k_{\text{eff}}$. For example, the formation of steam bubbles in a water coolant will decrease the amount of moderator in the core and thus alter $k_{\text{eff}}$.

No matter the mechanism that causes $k_{\text{eff}}$ to change, it is crucial that the resulting power transient be kept within strict limits so as to avoid core damage from excessive heat generation. The transient response and the manner in which $k_{\text{eff}}$ changes as a result of various core changes is thus of great importance for the safe design of a nuclear reactor.

In this chapter, kinetic equations are developed to describe how the neutron population in a core varies in time as $k_{\text{eff}}$ changes. Then various mechanisms which cause $k_{\text{eff}}$ to change as a result of power production are discussed.

#### 10.6.1 A Simple Reactor Kinetics Model

Consider a core in which the neutron cycle takes $l'$ seconds to complete. The change $\Delta n$ in the total number of thermal neutrons in one cycle at time $t$ is $(k_{\text{eff}} - 1)n(t)$,
where \( n(t) \) is the number of neutrons at the beginning of the cycle. Thus

\[
\Delta n(t) = \ell \frac{dn(t)}{dt} = (k_{\text{eff}} - 1)n(t)
\]

or equivalently

\[
\frac{dn(t)}{dt} = \frac{k_{\text{eff}} - 1}{\ell'} n(t).
\]

The solution of this first-order differential equation is

\[
n(t) = n(0) \exp \left[ \frac{k_{\text{eff}} - 1}{\ell'} t \right]
\]

where \( n(0) \) is the neutron population at \( t = 0 \). Notice that in this simple model, the neutron population (and hence the reactor power) varies exponentially in time if \( k_{\text{eff}} \neq 1 \).

However, this simple kinetics model is not very realistic. Consider, a reactor with \( \ell' = 10^{-4} \) s and operating at a steady-state level of \( n_0 \) neutrons. At \( t = 0 \), \( k_{\text{eff}} \) is increased slightly from 1 to 1.001 (a very small change). After 1 second, the neutron population, according to Eq. (10.12), would be

\[
n(1 \text{ s}) = n_0 \exp \left[ \frac{1.001 - 1}{10^{-4}} \right] = n_0 e^{10} \approx 20,000 n_0.
\]

Clearly such a reactor would be uncontrollable by a human operator!

### 10.6.2 Delayed Neutrons

If all fission neutrons were emitted at the time of fission, then the simple kinetics model of Eq. (10.11) would apply, and controllable nuclear reactors could not be built. Fortunately, a small fraction \( \beta \) (0.65\% for \(^{235}\text{U}\)) of fission neutrons are emitted, not during the fission event, but by the radioactive decay of daughters of certain fission products at times up to minutes after the fission event that created the fission products. The fission products, whose daughters decay by neutron emission, are called *delayed neutron precursors* and the emitted neutrons are called *delayed neutrons*. An example of a delayed neutron precursor is \(^{137}\text{I}\) whose decay chain, shown in Fig. 10.6, includes \(^{137}\text{Xe}\) which decays either by \( \beta^- \) or neutron emission. The half-life of \(^{137}\text{Xe}\) is exceedingly small, so that the apparent emission rate of the delayed neutron is determined by the half-life of the delayed neutron precursor \(^{137}\text{I}\).

There are many fission-products which lead to delayed-neutron emission. About twenty such nuclides have been identified; many others remain to be identified. Because of the importance of delayed neutrons in slowing down the neutron cycle,
Table 10.5. Half-lives and yield-fractions $\beta_i$ of a six delayed-neutron group representation for thermal fission of three important fissile nuclides. Here $\beta_i$ is the fraction of the $\nu$ fission neutrons emitted by the $i$th delayed neutron precursor group. Source: [Keepin 1965].

<table>
<thead>
<tr>
<th>Group</th>
<th>$^{235}$U</th>
<th>$^{233}$U</th>
<th>$^{239}$Pu</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Half-life (s)</td>
<td>fraction $\beta_i$</td>
<td>Half-life (s)</td>
</tr>
<tr>
<td>1</td>
<td>55.7</td>
<td>0.00021</td>
<td>55.0</td>
</tr>
<tr>
<td>2</td>
<td>22.7</td>
<td>0.00142</td>
<td>20.6</td>
</tr>
<tr>
<td>3</td>
<td>6.22</td>
<td>0.00127</td>
<td>5.00</td>
</tr>
<tr>
<td>4</td>
<td>2.30</td>
<td>0.00257</td>
<td>2.13</td>
</tr>
<tr>
<td>5</td>
<td>0.610</td>
<td>0.00075</td>
<td>0.615</td>
</tr>
<tr>
<td>6</td>
<td>0.230</td>
<td>0.00027</td>
<td>0.277</td>
</tr>
<tr>
<td>total</td>
<td>-</td>
<td>0.00065</td>
<td>-</td>
</tr>
</tbody>
</table>

Many experiments have been performed to identify the delayed-neutron yields and the rate at which they are emitted after the fission events. The results are grouped by the apparent half-lives of the observed emission rates i.e., delayed neutron precursors with similar half-lives are placed in the same delayed-neutron group. The delayed-neutron fraction or yield, $\beta_i$ for group $i$, is the fraction of the total fission neutrons that are eventually emitted by the decay of the $i$th type of delayed neutron precursor. Typically, six delayed-neutron groups are used in most transient calculations, although a single group approximation is often used for simplification of the kinetic equations. The yields and half-lives of a six delayed-neutron group model for thermal fission of the three fissile isotopes are shown in Table 10.5.

From the table, it is seen that the total delayed neutron fraction $\beta$ (the sum of the $\beta_i$) is considerably less for $^{233}$U and $^{239}$Pu than for $^{235}$U. In a power reactor, as $^{238}$U is converted to $^{239}$Pu and more and more power is generated from the fission of $^{239}$Pu, the delayed neutron fraction decreases and the reactor becomes more responsive to reactivity changes.

Delayed neutrons are emitted with a range of energies, the average energy being about one-half of that for prompt fission neutrons. As a consequence, delayed neutrons have less chance of leaking from the core as they slow down since they have less energy to lose before becoming thermal than do the more energetic prompt fission neutrons. As a consequence, the effective delayed neutron fraction $\bar{\beta}$, i.e., the delayed neutron fraction that would be needed if delayed neutrons were emitted with the same energy as prompt fission neutrons, is typically 10-15% larger than the physical delayed neutron fraction $\beta$.

10.6.3 Reactivity and Delta-k

Although the single parameter $k_{\text{eff}}$ clearly defines the criticality state of a reactor, the value of $k_{\text{eff}}$ for subcritical or supercritical reactors seldom varies by more than a few percent from its critical value of unity. Similarly, changes such as the movement of a control rod often produces changes in $k_{\text{eff}}$ only in the third or fourth significant figure. To emphasize the degree of departure from criticality, several slightly differ-
ent measures of how much $k_{\text{eff}}$ departs from the critical value of unity have come into wide use.

One such parameter called “delta-k” is defined simply as $\delta k \equiv k_{\text{eff}} - 1$. A closely related measure is the reactivity

$$\rho = \frac{k_{\text{eff}} - 1}{k_{\text{eff}}} = \frac{\delta k}{k_{\text{eff}}}.$$ \hspace{1cm} \text{(10.14)}

For low-power research reactors with little change in fuel composition over time, reactivity is usually measured in multiples of the delayed neutron fraction $\beta$ in units called “dollars”, i.e., $k($) = \rho/\beta$. Clearly any of the parameters $k_{\text{eff}}$, $\delta k$, $\rho$, and $k$ can be obtained from any other.

The great advantage of using $\delta k$, $\rho$ or $k$ is that the criticality state of the reactor is immediately apparent from the sign of these parameters. They are all positive for a supercritical system, negative for a subcritical reactor, and zero at criticality. The reactivity in dollars $k$ has the added advantage that the dangerous super prompt critical state (discussed in the next section) is readily recognized, i.e., when $k > 1$.

Example 10.7: A reactivity of 0.1$ is inserted into an initially critical uranium-fueled reactor. What is $k_{\text{eff}}$ of the reactor after the insertion? The reactivity in dollars is related to $k_{\text{eff}}$ by

$$k($) = \frac{\rho}{\beta} = \frac{k_{\text{eff}} - 1}{\beta k_{\text{eff}}}.$$

Solving for $k_{\text{eff}}$ we find

$$k_{\text{eff}} = \frac{1}{1 - \beta k($)} = \frac{1}{1 - 0.0065 \times 0.1} = 1.00065.$$

10.6.4 Revised Simplified Reactor Kinetics Models

Small Deviations from Criticality

The great importance of delayed neutrons is that they effectively slow down the neutron cycle time in reactors with $k_{\text{eff}}$ only slightly different from the critical value of unity. This lengthening of the neutron cycle time in turn causes the neutron population and reactor power to vary sufficiently slowly that control of the chain reaction is possible. Consider a thermal reactor fueled with $^{235}\text{U}$. The average half-life of the delayed neutron precursors is about $T_{1/2} = 8.8$ s so that their average lifetime is $\tau = T_{1/2}/\ln 2 \approx 12.8$ s. A fraction $\beta$ of the fission neutrons requires a cycle time of $\ell' + \tau$, while a fraction $(1 - \beta)$ is the prompt-neutron fraction and requires a cycle time of only $\ell'$. The average or effective generation time required for all the neutrons produced in a single neutron cycle is thus

$$\overline{\ell} = (1 - \beta)\ell' + \beta(\ell' + \tau) = \ell' + \beta\tau.$$ \hspace{1cm} \text{(10.15)}
Because $\ell' \leq 10^{-4}$ s and $\beta \tau = 0.0065 \times 12.8 = 0.083$ s, the effective generation time is $\bar{t} \approx \beta \tau$.

The simple reactor kinetics model developed in Section 10.6.1 thus must be modified by replacing the prompt generation lifetime $\ell'$ with the effective generation time $\bar{t}$. Again the neutron population varies exponentially in time as

$$n(t) = n_o \exp\left(\frac{k_{eg} - 1}{\bar{t}} t\right) = n_o \exp(t/T),$$

(10.16)

where the period or e-folding time is seen to be

$$T = \bar{t}/(k_{eg} - 1) = \bar{t}/\delta k = \beta \tau/\delta k.$$

(10.17)

This result is valid provided $|\delta k| << \beta$. Notice that this period for small departures from criticality is independent of the prompt generation lifetime $\ell'$.

Consider the example of Section 10.6.1 in which $\delta k = 0.001$. For this case, the period is $T = \beta \tau/\delta k = 0.0065 \times 12.8/0.001 = 83$ s, so that after one second the neutron population is

$$n(1 \text{ s}) = n_o \exp(1/83) = 1.012 n_o,$$

(10.18)

a much more manageable variation in neutron population.

**Super Prompt Critical Response**

As $k_{eg}$ increases past the critical value of unity, the reactor period becomes shorter and the neutron population increases more quickly. When $k_{eg}$ becomes greater than $1 + \beta$, the chain reaction can be sustained by the prompt neutrons alone. In such a super prompt critical reactor the delayed neutrons can be ignored. Then there is a prompt gain of $k_{eg} - 1 - \beta = \delta k - \beta > 0$ in a prompt cycle time of $\ell'$. The increase in the neutron population in one cycle is thus

$$\ell' \frac{dn(t)}{dt} = (\delta k - \beta)n(t).$$

(10.19)

The population thus increases exponentially as $n(t) = n_o \exp(t/T)$ where the period is

$$T = \ell'/(\delta k - \beta), \quad \delta k >> \beta.$$

(10.20)

For this extreme case, the reactor period is seen to be directly proportional to the prompt generation lifetime.

**Subcritical Reactors**

If $k_{eg}$ were reduced from its critical value of unity, the neutron population would decrease, and would do so more rapidly for larger reductions in $k_{eg}$. Although the neutron population produced by prompt fission neutrons can be made to decrease rapidly by a large reduction in $k_{eg}$, the delayed neutron population can decrease no faster than the decay rate of the delayed neutron precursors. After a few minutes following a large decrease in $k_{eg}$, the neutron population decreases at the rate
of decay of the longest lived delayed neutron precursor which has a half-life of about 55 s (see Table 10.5) or a mean lifetime of $\ln 2/T_{1/2} \approx 80$ s. Thus, even if all control rods of a reactor were inserted to shut the reactor down, the neutron population, and hence the power, would after a short time decrease exponentially with a period of -80 s. This inability to quickly reduce a reactor’s power, as well as the decay heat produced by the radioactive decay of fission products in the fuel, require power reactors to maintain cooling of the core for long periods following a reactor shutdown.

**Reactor Kinetic Models with Delayed Neutrons**

The above cases are for extreme changes in $k_{eff}$. To obtain the kinetic response of a reactor to an arbitrary variation of $k_{eff}$, it is necessary to describe the dynamic variation of the delayed neutron precursors as well as the neutrons in a core. Such a treatment leads to a set of coupled first-order differential equations, one for each delayed neutron precursor group and one for the neutron population or power of the reactor. The derivation of these so-called *point reactor kinetic equations* is provided in Addendum 2 of this chapter. These point reactor kinetic equations have been found to describe very accurately the time variation of the neutron population in a reactor caused by changes in $k_{eff}$.

**Example 10.8:** If the reactor of Example 10.7 was initially operating at a power of 10 W. How long after the 0.1$ reactivity insertion is it before the reactor power reaches 10 kW? From Example 10.7, the reactivity insertion is $\delta k = 0.00065$ and from Eq. (10.17) we find the resulting reactor period is

$$T = \frac{\beta k}{\delta k} = \frac{0.00065 \times 12.8 \text{ s}}{0.00065} = 128 \text{ s}.$$  

Since the reactor power $P(t)$ is proportional to the neutron population $n(t)$, we have from Eq. (10.16), $P(t) = P(0) \exp(t/T)$. Solving for $t$, we find

$$t = T \ln\left[\frac{P(t)}{P(0)}\right] = 128 \ln(10,000/10) = 884 \text{ s} = 14.7 \text{ min}.$$  

**10.6.5 Power Transients Following a Reactivity Insertion**

The response of a reactor to a change in $k_{eff}$ is of great importance in the design of a reactor. For example, the rate at which power can be decreased in an emergency situation dictates the design of several complex systems to maintain core cooling in an emergency. Likewise, safety features must be designed into a reactor to insure than $k_{eff}$ cannot become so much greater than unity that rapid and uncontrollable power increases occur. In this section, the power transients following a reactivity insertion into the core are considered.

**Steady-State Operation**

One of the most important design and operational problems is the transient caused by the insertion of reactivity (positive or negative) into a reactor operating at some steady-state neutron level $n_0$. For a reactor in which there are negligible neutron
sources that are independent of the neutron field, steady-state power occurs only if the reactivity is zero \((k_{en} = 1)\).

However, at low power levels \((\gamma, n)\) or \((\alpha, n)\) reactions produce a background source of neutrons. Let the steady-rate of production of neutrons by sources that do not depend on the neutron field be denoted by \(S_0\). In steady-state, the time derivatives in the point reactor kinetic equation, Eqs. (10.71), vanish and the following steady-state value of the neutron population is obtained:

\[
n_0 = \frac{\ell}{\beta k_o} S_0,
\]

where the effective neutron lifetime \(\ell = \ell' / k_{en} \approx \ell'\). Notice from this last result, that, for a reactor in steady state with an non-fission source \(S_0\), the reactivity must necessarily be negative (i.e., the reactor is subcritical). As \(S_0 \rightarrow 0\), \(k_o\) must also vanish (i.e., the reactor becomes critical) to maintain a steady-state neutron population.

**Response to a Step Reactivity Insertion**

When the reactivity of a reactor varies about its critical value of zero, the neutron population, and hence the power, varies according to the point reactor kinetics equations, Eqs. (10.70) or Eqs. (10.71). One of the most fundamental kinetics problems is the transient produced by a constant reactivity insertion (positive or negative) into a source-free, critical reactor operating at a steady neutron level \(n_0\). In particular, at \(t = 0\) the reactivity is changed from its steady-state value of zero to \(k_o\), i.e.,

\[
k(t) = \begin{cases} 
  0 & t < 0 \\
  k_o & t \geq 0
\end{cases}.
\]

If \(k_o\) were positive, we would expect the neutron level to rise; by contrast, for negative \(k_o\), we would expect the level to decrease and eventually vanish. Example transients are shown in Fig. 10.7. As seen from this figure, the neutron level (or power) experiences a rapid *prompt jump* followed by an exponential variation (the straight lines in the semi-log plot). The time required for the neutron level to change by a factor of \(e\) is called the *reactor period*. Of great operational importance is the period of the asymptotic neutron variation, i.e., the slope of the transient on a semi-log plot, at long times after the reactivity insertion.

For negative insertions, the initial prompt drop arises from the decrease in prompt neutrons in each succeeding generation. However, the decay of the existing delayed neutron precursors acts as a neutron source which decays more slowly with the half-lives of the precursors. Even for very large negative reactivity insertions, the neutron level after the prompt jump cannot decrease any faster than the longest lived precursor group (with about a 55 second half-life, or about 80 seconds to decrease by a factor of \(e\)). This physical limit on how quickly a reactor can be shut down is a major safety concern and necessitates that considerable attention be given to emergency core cooling systems.

For a positive reactivity insertion, there is an initial rapid rise in the neutron population because of the increased multiplication of the prompt neutrons. For \(k_o < 1\), however, there is less than one prompt neutron produced per generation, and hence there must be delayed neutrons introduced to keep the system supercritical.
i.e., we must wait for the decay of newly created delayed neutron precursors, thereby slowing down the rate of rise in the neutron level. However, if $k_0 > 1$, the neutron population is self-sustaining on prompt neutrons alone, and the rate of increase in the neutron level is very rapid, limited by only the neutron generation lifetime.

Since the point reactor kinetics equations, are first-order differential equations, we expect the mathematical solution for $n(t)$ and for the delayed neutron precursor densities $C_i(t)$ to be of the form $e^{\omega t}$. If we substitute this assumed form into Eqs. (10.71), we find that the only permissible values of $\omega$ must satisfy

$$\omega \left[ \frac{\ell}{\beta} + \sum_{i=1}^{G} \frac{a_i}{\lambda_i + \omega} \right] = k_0.$$  \hspace{1cm} (10.23)

This equation is known as the *inhour equation* since the units of $\omega$ are inverse time, e.g., h\(^{-1}\). In this equation, $\lambda_i$ is the decay constant for the $i$th delayed neutron precursor group, $a_i \equiv \beta_i / \beta$, and $G$ is the number of delayed neutron precursor groups. There are $G + 1$ solutions of this equation, whose values are denoted by $\omega_j$, $j = 0, \ldots, G$ where we order the solutions such that $\omega_0 > \omega_1 > \ldots > \omega_G$.  

---

*Figure 10.7.* Variation of neutron level caused by various step reactivity insertions into a $^{235}$U fueled reactor with $\beta = 0.007$ and $\ell = 0.0004$.  

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Thus the most general solution of Eqs. (10.71) is of the form

\[ n(t) = \sum_{j=0}^{G} A_j e^{\omega_j t} \]  

(10.24)

where \( A_j \) are constants. A rigorous derivation for a one delayed-neutron group model is given in Addendum 3 to this chapter. It can be shown that the largest root of the inhour equation, \( \omega_0 \), is positive if \( k_0 > 0 \) and negative if \( k_0 < 0 \). The other \( \omega_j \) are always negative regardless of \( k_0 \). The \( e^{\omega_j t} \), \( j = 1, \ldots, G \) terms decay away more rapidly than the \( e^{\omega_0 t} \) term. Thus at long times after the reactivity insertion

\[ n(t) = A_0 \exp[\omega_0 t] = A_0 \exp[t/T_{as}] \]  

(10.25)

where the asymptotic period \( T_{as} \equiv 1/\omega_0 \). Hence we see that the asymptotic (large time) neutron variation is purely exponential with a period (or e-folding time) of \( T_{as} \).

The asymptotic period as a function of the reactivity insertion is shown in Figs. 10.8 and 10.9. Notice that the asymptotic period is almost independent of the neutron lifetime \( \ell \) for \(-\infty < k_0 < 1\). For small reactivity insertions (\(|k_0| \leq 0.1\)), it can be shown that

\[ T_{as} \approx \ell/(\beta k_0) \]  

(10.26)

which is in agreement with Eq. (10.17). However, as \( k_0 \) becomes greater than 1, the asymptotic period changes dramatically and becomes proportional to the neutron lifetime. It can be shown that for \( k_0 >> 1 \), \( T_{as} \approx (\ell/\beta)/(k_0 - 1) \), which is in agreement with Eq. (10.20).
10.7 Reactivity Feedback

The energy and nuclear reactions that occur during the operation of a nuclear reactor change the material properties of the core and thus the multiplication factor $k_{\text{eff}}$. This change in the reactivity of the reactor is called feedback reactivity and is illustrated schematically in Fig. 10.10.

![Figure 10.10](neutronics.png)

Figure 10.10. The total reactivity driving the neutron cycle is the sum of externally applied reactivity $k_{\text{ex}}$ and the feedback reactivity $k_f$.

The physical causes of feedback reactivity can be grouped into two categories: reactivities resulting from isotopic changes and those from temperature changes. Some of the most common feedback mechanisms are discussed below. In general, the amount of feedback reactivity $k_f(t)$ at any time $t$ depends on the entire past power history of the reactor, i.e., on $n(t')$, $t' < t$.

10.7.1 Feedback Caused by Isotopic Changes

As a reactor generates power, the neutron field causes changes in the isotopic composition of the core material and thus in $k_{\text{eff}}$. Reactivity transients produced by isotopic changes are usually very slow, occurring over a period of hours at the fastest to years. Some of the important isotopic changes are described below.

Fuel Burnup

To sustain the neutron chain reaction, the fissile fuel must be consumed. Through this fuel consumption, $n_f$ decreases slowly in time and causes a negative reactivity feedback which must be compensated for by removing other negative reactivity from the core. This compensation for fuel burnup can be accomplished in a variety of ways, e.g., by diluting the boric acid concentration in the water moderator, by withdrawing control rods, or replacing high-burnup fuel with fresh fuel.

Fuel Breeding

In any reactor there are fertile isotopes which absorb neutrons, and through subsequent radioactive decays, produce atoms of fissile fuel. For example, in $^{235}\text{U}$-fueled reactors, the $^{238}\text{U}$ atoms can absorb neutrons and, after two radioactive decays, yield fissile $^{239}\text{Pu}$. This breeding mechanism causes a positive reactivity feedback, and is an important feedback mechanism in reactors with low enrichments, particularly in reactors fueled with natural uranium. In LWRs with typically 2-3% enrichments, at the end of a fuel-cycle (the time at which the reactor must be refueled) almost as much power is generated from the fission of the bred $^{239}\text{Pu}$ as from $^{235}\text{U}$.
**Fission Product Poisons**

As the fission products and their daughters accumulate in a reactor, these new isotopes may also absorb neutrons and decrease the thermal utilization factor $f$. Two particular fission product poisons, $^{135}$Xe and $^{149}$Sm have particularly large absorption cross sections for thermal neutrons and can have severe negative reactivity effects on reactor operations. These fission-product poisons are discussed in detail in the next section.

**Burnable Poisons**

In many power reactors, small amounts of material with large absorption cross sections (e.g., samarium or gadolinium in fuel rods, or boric acid in a water coolant) are incorporated into the core material. Such material initially decreases $f$, but during reactor operation it is consumed and, thereby, increases $f$ to offset the decrease in $\eta f$ caused by fuel consumption.

10.7.2 Feedback Caused by Temperature Changes

The temperature of the core material affects the rate at which neutrons interact with the material. These changes in interaction rates arise from changes in atomic concentrations, reaction probabilities, and geometry changes. The reactivity transients caused by temperature changes are usually very fast, as fast as the temperature itself changes. Crucial to safe reactor operation is a reactor design that has an overall negative reactivity feedback as the core temperature increases. Some of the important mechanisms are discussed below.

**Changes in Atomic Concentrations**

Most materials expand as their temperature increases, thereby decreasing their atomic concentration and their macroscopic cross sections. Most LWRs are slightly undermoderated, i.e., the moderator-fuel ratio is less than the ratio that produces a maximum in $k_{\infty}$. Thus as the moderator/coolant heats and expands, some is expelled from the core, making the reactor even more undermoderated and thus decreases $k_{\text{eff}}$. An even more severe change in atom densities is caused by the formation of steam bubbles. Steam generation expels liquid from the core and introduces a large negative reactivity. In solid graphite moderated cores, the expansion of the moderator decreases the thermalization of neutrons and thus also causes a negative reactivity feedback.

In most reactors the fuel is in a ceramic form that expands very little with increasing fuel temperature, and, consequently, the negative fuel expansion reactivity effect is very small in most reactors. Only in small experimental reactors composed of fissile metal, does the fuel expansion effect cause a large negative reactivity.

**Changes in the Neutron Energy Distribution**

As the temperature of the core material increases, the thermal neutrons maintain a Maxwellian energy distribution one but shifted increasingly towards higher energies. The thermal spectrum is said to harden. If all the cross sections in the core had exactly a $1/\nu$ dependence, there would be no change in the thermal-neutron interaction rates. However, most heavy nuclides have resonances near the upper end of the thermal energy range and their cross sections, consequently, are not exactly $1/\nu$ in the thermal energy region. For example $^{239}$Pu has a large resonance at about...
0.3 eV, and, as the thermal neutrons shift in energy towards this resonance, more neutrons are absorbed by $^{239}$Pu and thus the fission rate increases, thus producing a positive reactivity feedback. By contrast, hardening the thermal neutron spectrum causes a very slight decrease in absorption by $^{235}$U.

This hardening of the thermal neutron spectrum with increasing temperature is taken to an extreme in the TRIGA class of reactors which use as fuel enriched $^{235}$U blended in zirconium hydride. As the fuel temperature increases, vibrating hydrogen atoms trapped in the zirconium-hydride crystal lattice can transfer some of their vibrational energy (about 0.13 eV) to thermal neutrons, thereby removing them from the thermal energy region so they are less likely to be absorbed by the fuel. This very rapid negative reactivity feedback effect is the reason these reactors can be operated in a pulse mode, in which a large positive reactivity is inserted into the core by rapidly removing a control rod to make the reactor very super prompt critical. However, the ZrH negative temperature feedback acts within a few ms to stop the runaway chain reaction, which has increased the reactor power by many thousands of times the initial power, and brings the reactor power back to safe limits.

**Changes in Resonance Interactions**

The most important negative temperature feedback mechanism in most reactors is that provided by the change in neutron interaction rates with materials having large cross section resonances just above the thermal region. Only epithermal neutrons (neutrons with energies above the thermal region) with energies equal to the resonance energy interact with the fuel. However, increasing the fuel temperature causes the fuel atoms to move more rapidly and the relative energy between a neutron and a fuel atom changes. This spreading out of the relative kinetic energy allows neutrons with energies near the resonance energy to interact with the fuel atoms, thereby, increasing the neutron interaction rate. The effective cross section is said to be *Doppler broadened*, and allows more neutrons to interact with the fuel.

This Doppler effect is the dominant feedback mechanism in many thermal reactors with low enrichment fuel. As the fuel temperature increases, more thermalizing neutrons are absorbed in the $^{238}$U resonances as the neutrons slow down. The resonance escape probability $p$ thus decreases causing a rapid negative reactivity feedback. However, an increase in fuel temperature also causes an increase in neutron absorption by the fissile atoms (thus increasing $f$), particularly by $^{239}$Pu which has a very low energy resonance at 0.3 eV. This positive reactivity feedback places a restriction on how much $^{239}$Pu can be used in a reactor, since the overall reactivity feedback must be negative.

**Changes in Geometry**

When thermal gradients are produced in a core, the fuel rods undergo differential expansions resulting in, for example, bowing, which changes the core geometry slightly and produces small positive or negative reactivity changes. A more severe geometry change is that caused by an accident condition in which the fuel undergoes unexpected deformations, melting, or even fuel dispersal. Reactivity changes produced by core disruptions are a major consideration in safety analyses.
10.8 Fission Product Poisons

The fission products that accumulate in a reactor core are of concern for two reasons. First, they act as long-term heat sources through their radioactive decays. Second, they act as parasitic neutron absorbers or poisons that, over time, decrease the thermal utilization factor \( f \) and, thus, introduce negative reactivity into a core.

The reactivity \( \rho_p \) introduced by a fission product poison is directly proportional to its average concentration \( N_p \) in the core. To see this, we start with

\[
\rho_p = \frac{k'_{\text{eff}} - 1}{k'_{\text{eff}}} - \frac{k_{\text{eff}} - 1}{k_{\text{eff}}}
\]

(10.27)

where \( k'_{\text{eff}} \) indicates the core with the poison included and \( k_{\text{eff}} \) refers to the same core without the poison. Since the poison changes only the thermal utilization factor, the two multiplication factors are related to each other by \( k'_{\text{eff}} = k_{\text{eff}} f' / f \). If we assume the unpoisoned core is critical \( (k_{\text{eff}} = 1) \), Eq. (10.27) becomes

\[
\rho_p = \frac{k'_{\text{eff}} - 1}{k_{\text{eff}}} = 1 - \frac{1}{k_{\text{eff}}} = 1 - \frac{f' - 1}{f'} k_{\text{eff}} = 1 - \frac{f' - 1}{f'}
\]

\[
1 - \frac{\Sigma\Sigma' / \Sigma}{\Sigma_a} = -\frac{\Sigma\Sigma' / \Sigma}{\Sigma_a} = -\frac{\Sigma\Sigma' / \Sigma}{\Sigma_a} = -\sigma_p \Sigma_a N_p \Sigma / \Sigma_f.
\]

(10.28)

where \( \sigma_p \) is the thermal microscopic absorption cross section for the poison.

The ratio \( \Sigma_f / \Sigma_a = (\Sigma_f / \Sigma_a) (\Sigma_a / \Sigma_a) = (\eta / \nu) f \). For most light water reactors, fuel enriched in \( ^{235}\text{U} \) to about 2.5% is used so that \( \eta \approx 1.8 \) and \( \nu = 2.43 \). For these reactors \( f \approx 0.8 \), so that \( \Sigma_f / \Sigma_a \approx 0.6 \). The poison reactivity is given by

\[
\rho_p \approx -0.6 \sigma_p N_p / \Sigma_f.
\]

(10.29)

Thus, to determine the reactivity transient caused by a particular fission product poison, we need to find \( N_p(t) / \Sigma_f \), a quantity that is found from the decay and buildup equations for the poison decay chain.

There are hundreds of different fission products and their decay daughters that are created in a core over time, each with a different absorption cross section for thermal neutrons. With the exception of \( ^{135}\text{Xe} \) and \( ^{149}\text{Sm} \), which have unusually large cross sections for absorbing a thermal neutron, it is impractical to keep track of the individual isotopes separately. Rather, the fission products and their daughters are treated collectively. For fission products produced from the fission of \( ^{235}\text{U} \), it is often assumed that each fission produces 1 atom of stable poison with an absorption cross section of 50 barns. While this simplistic rule-of-thumb works for long-term calculations of burnup effects, the two particular poison cases \( ^{135}\text{Xe} \) and \( ^{149}\text{Sm} \) have such large absorption cross sections that they must be treated separately.

10.8.1 Xenon Poisoning

\( ^{135}\text{Xe} \) is a member of the fission product decay chain shown below. It is of importance because it has the largest thermal neutron absorption cross section of all
isotopes, namely, $\sigma_a^X = 2.7 \times 10^6$ barns. Thus a very small atomic concentration of this nuclide can have a considerable reactivity effect.

The isotopes $^{135}\text{Te}$, $^{135}\text{I}$ and $^{135}\text{Xe}$ are all produced as fission products. However, because of the very short half-life of $^{135}\text{Te}$ compared to the other members of the chain, it is usually assumed that $^{135}\text{Te}$ immediately decays to $^{135}\text{I}$. The fission product yield $\gamma$ of $^{135}\text{I}$ for $^{235}\text{U}$ (including that of $^{135}\text{Te}$) is 0.061 and, for $^{135}\text{Xe}$, $\gamma_X = 0.003$. The decay constants for $^{135}\text{I}$ and $^{135}\text{Xe}$ are $2.9 \times 10^{-5}$ s$^{-1}$ and $2.1 \times 10^{-5}$ s$^{-1}$, respectively.

The production rate (per unit volume) of $^{135}\text{I}$ as a fission product is $\gamma \Sigma_f \phi(t)$ where $\phi(t)$ is the average thermal flux density in the core at time $t$. Its disappearance rate through radioactive decay is $-\lambda I(t)$ where $I(t)$ is the average $^{135}\text{I}$ atomic concentration in the core. The absorption cross section for $^{135}\text{I}$ is negligible and burnup of $^{135}\text{I}$ does not contribute to its disappearance. For $^{135}\text{Xe}$, the production rate is by the radioactive decay of $^{135}\text{I}$ and as a fission product, namely, $\lambda I(t) + \gamma_X \Sigma_f \phi(t)$. $^{135}\text{Xe}$ disappears by radioactive decay and by neutron absorption at a rate $-\lambda_X X(t) - \sigma_X^X \phi(t) X(t)$, where $X(t)$ is the average atomic concentration of $^{135}\text{Xe}$ in the core at time $t$. Thus the decay/buildup equations for $^{135}\text{I}$ and $^{135}\text{Xe}$ are

\begin{align*}
\frac{dI(t)}{dt} &= -\lambda I(t) + \gamma \Sigma_f \phi(t) \quad (10.30) \\
\frac{dX(t)}{dt} &= \lambda I(t) + \gamma_X \Sigma_f \phi(t) - \lambda_X X(t) - \sigma_X^X \phi(t) X(t). \quad (10.31)
\end{align*}

The solution of these coupled differential equations subject to arbitrary initial values $I(0)$ and $X(0)$ and for a constant flux density $\phi_o$ is

\begin{align*}
\frac{I(t)}{\Sigma_f} &= \frac{I(0)}{\Sigma_f} \exp[-\lambda t] + \frac{\gamma \phi_o}{\lambda} \{1 - \exp[-\lambda t]\} \quad (10.32) \\
\frac{X(t)}{\Sigma_f} &= \frac{X(0)}{\Sigma_f} \exp[-(\lambda_X + \sigma_X^X \phi_o)t] + \frac{(\gamma + \gamma_X) \phi_o}{\lambda_X + \sigma_X^X \phi_o} \{1 - \exp[-(\lambda_X + \sigma_X^X \phi_o)t]\} \\
&\quad + \frac{(\lambda I(0)/\Sigma_f) - \gamma_X \phi_o}{\lambda_X - \lambda + \sigma_X^X \phi_o} \{\exp[-\lambda_X t] - \exp[-(\lambda_X + \sigma_X^X \phi_o)t]\}. \quad (10.33)
\end{align*}

**Equilibrium Xenon Poisoning**

For a reactor operating at a constant flux density $\phi_o$, the equilibrium concentrations of $^{135}\text{I}$ and $^{135}\text{Xe}$ are found from Eqs. (10.30) and (10.31) by setting the time derivative to zero. The result is

\begin{align*}
I_o &= \frac{\gamma \Sigma_f \phi_o}{\lambda} \quad (10.34) \\
X_o &= \frac{(\gamma + \gamma_X) \Sigma_f \phi_o}{\lambda_X + \sigma_X^X \phi_o} \quad (10.35)
\end{align*}
The equilibrium concentrations are shown in Fig. (10.11). Notice that, while the $^{135}$Xe concentration is independent of $\phi_0$ at high flux density levels, the $^{135}$I concentration continues to increase linearly with $\phi_0$. This has profound consequences for reactors operating at high power.

![Figure 10.11. Equilibrium $^{135}$I and $^{135}$Xe concentrations as a function of the steady-state flux density $\phi_0$.](image1)

![Figure 10.12. $^{135}$Xe transients following shutdowns from equilibrium at various constant flux densities.](image2)

![Figure 10.13. $^{135}$Xe transient for the buildup to equilibrium, the transient following shutdown from equilibrium, and three transients (dashed lines) following restart during the shutdown transient.](image3)

![Figure 10.14. $^{135}$Xe transient following a shutdown from equilibrium showing the time-to-poison and the poison shutdown time. Equilibrium flux density before shutdown is $\phi_0 = 10^{14}$ cm$^{-2}$ s$^{-1}$.](image4)

**Transient Following Shutdown from Equilibrium**

In reactors that operate at high power, $\phi_0 > 10^{13}$ cm$^{-2}$ s$^{-1}$, there are higher equilibrium levels of $^{135}$I (potential $^{135}$Xe) than of $^{135}$Xe, which has a slower decay rate. Thus, if the reactor should suddenly shut down, the $^{135}$I would decay to $^{135}$Xe faster than the $^{135}$Xe could decay away, producing initially an increase in the $^{135}$Xe concentration. Eventually, the $^{135}$I would decay away, and the $^{135}$Xe concentration...
would finally begin to decrease as it decays. Fig. (10.12) shows the buildup of $^{135}\text{Xe}$ following the shutdown from various flux levels.

If during the shutdown transient, the reactor were started up again, the large absorption cross section for $^{135}\text{Xe}$ would cause this nuclide to be burned up very rapidly, reducing the xenon reactivity temporarily to below its equilibrium values. Examples of these restart transients are shown in Fig. (10.13).

One consequence of the large increase in the negative $^{135}\text{Xe}$ reactivity following a shutdown from equilibrium, is that very quickly the xenon reactivity may become greater than offsetting positive reactivity available by removing all control rods from the core. Should this occur, it is impossible to restart the reactor, and the reactor is said to have poisoned out. The time from the shutdown until the reactor poisons out is the called the time-to-poison (see Fig. (10.14)). Once the reactor has poisoned out, it is necessary to wait until the negative $^{135}\text{Xe}$ reactivity has peaked and descended back to a level that can be offset by all controllable positive reactivities. The time interval during which the reactor cannot be restarted is called the poison shutdown time and is typically of 15–25 hours duration. In many power or propulsion reactors, the time-to-poison is usually only a few tens of minutes, and the operator may experience considerable pressure to get the reactor restarted before it poisons out so as to avoid a lengthy period of lost production.

When the reactor is restarted during a shutdown transient, there is a rapid burnoff of $^{135}\text{Xe}$, and since, most of the $^{135}\text{I}$ has already decayed away, the negative xenon reactivity quickly drops below equilibrium levels (see the dashed lines in Fig. 10.13). However, over the next day of operation, the $^{135}\text{Xe}$ resumes its equilibrium level.

**Xenon Spatial Oscillations**

$^{135}\text{Xe}$ transients generally vary so slowly that they are usually easily controlled by reactor operators. However, in very large reactors with a small negative temperature reactivity coefficient, $^{135}\text{Xe}$ can induce a very complex instability in which the maximum of the flux density moves around the core with periods of about ten hours. The phenomenological sequence of events is as follows:

1. An initial asymmetry in the core flux distribution (e.g., caused by a control rod misalignment) causes a change in the $^{135}\text{I}$ buildup and $^{135}\text{Xe}$ burnup.

2. In the high-flux region, increased $^{135}\text{Xe}$ burnup initially allows the flux to increase even further causing an increase in $^{135}\text{I}$ production. Meanwhile in the low-flux region, $^{135}\text{I}$ decays to $^{135}\text{Xe}$ causing the flux to be even further depressed while the $^{135}\text{I}$ continues to decay.

3. Eventually, in the high-flux region, the $^{135}\text{I}$ reaches a new higher equilibrium, and its decay to $^{135}\text{Xe}$ causes the $^{135}\text{Xe}$ concentration to begin to rise, thereby reversing the pattern. Likewise, in the low-flux region, the $^{135}\text{Xe}$ becomes depleted by decay thereby allowing the flux to start to rise.

4. Repetition of this pattern, causes the flux profile to oscillate spatially with a period of around 10 hours.

These xenon spatial oscillations, while producing little overall reactivity change for the core, can cause the local flux and power density to vary by factors of three.
To avoid oscillations in reactors that are susceptible to them, it is necessary to implement a very complex series of control rod adjustments which are usually determined by a computer.

10.8.2 Samarium Poisoning

The second fission product poison which must be accounted for explicitly in power reactors is $^{149}\text{Sm}$. This stable nuclide is a daughter of the fission products $^{149}\text{Nd}$ and $^{149}\text{Pm}$. The decay chain for this nuclide is shown below.

$^{149}\text{Nd} \xrightarrow{\beta^-_{1.7\text{h}}} ^{149}\text{Pm} \xrightarrow{\beta^-_{53\text{h}}} ^{149}\text{Sm} \text{(stable)}$

Because the half-life of $^{149}\text{Nd}$ is so small compared to that of its daughter $^{149}\text{Pm}$, it is usually assumed to decay immediately to $^{149}\text{Pm}$. Thus the effective fission product yield $\gamma_\nu$ for $^{149}\text{Pm}$ is the sum of the actual yields for $^{149}\text{Nd}$ and $^{149}\text{Pm}$. For $^{235}\text{U}$, $\gamma_\nu = 0.0113$. Thus the production rate of $^{149}\text{Pm}$ (per unit volume) is $\gamma_\nu \Sigma_f \phi(t)$ and the rate of decay is $\lambda_P(t)$ where $P(t)$ is the core-averaged $^{149}\text{Pm}$ atomic concentration. The generation rate of $^{149}\text{Sm}$ is the decay rate of $^{149}\text{Pm}$. There is negligible production of $^{149}\text{Sm}$ as a direct fission product. Since $^{149}\text{Sm}$ is stable, the only way it can vanish is for it to absorb a neutron which it does at a volumetric rate of $\sigma_a^S \phi(t) S(t)$ where $S(t)$ is the average $^{149}\text{Sm}$ concentration. Thus the decay/buildup equations for $^{149}\text{Pm}$ and $^{149}\text{Sm}$ are

$$\frac{dP(t)}{dt} = -\lambda_P(t) + \gamma_\nu \Sigma_f \phi(t)$$

$$\frac{dS(t)}{dt} = \lambda_P(t) - \lambda_P \sigma_a^S \phi(t) S(t).$$

The solution of these coupled differential equations subject to arbitrary initial values $P(0)$ and $S(0)$ and for a constant flux density $\phi_\nu$ is

$$\frac{P(t)}{\Sigma_f} = \frac{P(0)}{\Sigma_f} \exp[-\lambda_P t] + \frac{\gamma_\nu \phi_\nu}{\lambda_P} \{1 - \exp[-\lambda_P t]\}$$

$$\frac{S(t)}{\Sigma_f} = \frac{S(0)}{\Sigma_f} \exp[-\sigma_a^S \phi_\nu t] + \frac{\gamma_\nu \phi_\nu}{\sigma_a^S \phi_\nu} \{1 - \exp[-\sigma_a^S \phi_\nu t]\}$$

$$+ \frac{(\lambda_P \phi_\nu S(0)/\Sigma_f) - \gamma_\nu \phi_\nu}{\sigma_a^S \phi_\nu - \lambda_P} \{\exp[-\lambda_P t] - \exp[-\sigma_a^S \phi_\nu t]\}. \quad (10.39)$$

Because of the long half-lives of $^{149}\text{Pm}$ and $^{149}\text{Sm}$, the buildup of $^{149}\text{Sm}$ to its equilibrium level takes many tens of hours, especially for reactors operating at low average flux densities $\phi_\nu$. From Eq. (10.36), it is seen the equilibrium $^{149}\text{Sm}$ is $S_o = (\gamma_\nu \Sigma_f)/\sigma_a^S$, a level that is independent of the flux density. Thus at equilibrium, all reactors have the same amount of $^{149}\text{Sm}$ poisoning. This buildup to equilibrium is shown in Figs. (10.15) and (10.16).

Following a shutdown from equilibrium, $^{149}\text{Sm}$ increases as the $^{149}\text{Pm}$ decays. This buildup of negative reactivity can be especially severe for reactors operating...
at high flux densities. However, unlike the $^{135}\text{Xe}$ transient following a reactor shutdown, the $^{149}\text{Sm}$ transient does not decay away with time since $^{149}\text{Pm}$ is stable. If a reactor does not have sufficient positive reactivity to compensate for this buildup following a scram, the reactor will permanently poison out and cannot be restarted until additional positive reactivity is added (e.g., by adding fresh fuel).

10.9 Addendum 1: The Diffusion Equation

To calculate the power distribution $P(r)$ in a reactor core, it is first necessary to know the distribution of the neutron flux density $\phi(r)$. If $E_r$ is the recoverable energy per fission (roughly 200 MeV), these two distributions are related by

$$P(r) = E_r \Sigma_f(r) \phi(r) \quad \text{MeV cm}^{-3} \text{s}^{-1}. \quad (10.40)$$

To calculate the flux density, it is necessary to solve the neutron diffusion equation. This equation is simply a mathematical relation balancing the rate of neutron loss and production at every point in the core. In this section, we derive the diffusion equation for the special case of one-speed neutrons diffusing through a homogeneous material. Such one-speed neutrons are descriptive of the thermal neutron population in a thermal reactor.

Consider an arbitrary volume $\Delta V$ in some medium through which one-speed neutrons diffuse by scattering randomly from the medium’s nuclei. Under steady-state conditions, the rate of neutron gain must equal the rate of neutron loss from the volume $\Delta V$, i.e.,

$$\text{leakage rate} + \text{absorption rate} = \text{production rate} \quad (10.41)$$

To obtain the diffusion equation, we simply have to express each term mathematically.
As neutrons scatter repeatedly, they change directions in each scatter, and thus perform a random walk as they move through the medium. In a medium containing many neutrons, each moving and scattering randomly, more neutrons will, on average, tend to move from regions of high concentration to regions of lower concentration than in the reverse direction. To express this mathematically, consider the flux profile shown in Fig. 10.17. The neutron density (and hence $\phi$) is higher to the left than to the right of the unit area perpendicular to the $x$-axis. It is reasonable to assume that the net flow across this unit area in a unit time, $J_x$, is proportional to the difference between the neutron densities on the low density side to that on the high density side. In other words, the net flow should be proportional to the negative slope of the neutron density or flux at the unit area, i.e., $J_x$ is proportional to $-[d\phi/dx]$. Thus we assume

$$J_x = -D\frac{d\phi}{dx},$$

(10.42)

where $D$ is the constant of proportionality known as the diffusion coefficient. This equation was first proposed by Adolf Fick in 1855 to describe the diffusion of randomly moving molecules in a fluid, and is consequently known as Fick's law. This empirical "law" is reminiscent of an earlier 1823 empirical observation by J.J.B. Fourier describing the net flow of heat energy (a manifestation of the random motion of a medium's atoms), namely the heat flow $= -k\,dT/dx$ where $k$ the is so-called thermal conductivity of the medium.

![Figure 10.17](image1.png)

**Figure 10.17.** The net flow of neutrons $J_x$ is away from the region of higher concentration.

![Figure 10.18](image2.png)

**Figure 10.18.** The leakage from the differential slab is the difference between the net flows at the two faces.

We now use Fick's law to express the leakage term in the balance relation of Eq. (10.41). Consider the differential slab of width $dx$ shown in Fig. 10.18. The net leakage rate from this slab is

$$\text{net leakage rate} = [J_x(x + dx) - J_x(x)]A$$

(10.43)

where $A$ is the area of the slab surfaces (in the $y$-$z$ plane). If we assume that the flux density $\phi(x)$ does not vary in the $y$ or $z$ directions, there would be no net leakage in these directions.
The rate of absorption of neutrons in the differential slab is the absorption rate per unit volume ($\Sigma_a \phi$) times the volume of the slab $\Delta V = A dx$, i.e.,

$$\text{absorption rate} = \Sigma_a \phi(x) A dx.$$  \hfill (10.44)

Neutrons are produced by fission ($\nu \Sigma_f \phi$ per unit volume per unit time) and by any "external" sources, such as ($\gamma$,n) or ($\alpha$,n) reactions, which are independent of the neutron field. We define $Q$ as the rate of neutron production per unit volume by such external sources. Then the rate of neutron production is

$$\text{production rate} = [\nu \Sigma_f \phi(x) + Q(x)] A dx.$$  \hfill (10.45)

Substitution of Eqs. (10.43), (10.44), and (10.45) into the balance relation Eq. (10.41) and simplification of the result give

$$\frac{J_x(x + dx) - J_x(x)}{dx} + \Sigma_a \phi(x) = \nu \Sigma_f \phi(x) + Q(x)$$

or

$$\frac{dJ_x(x)}{dx} + \Sigma_a \phi(x) = \nu \Sigma_f \phi(x) + Q(x).$$  \hfill (10.46)

Finally, with $J_x$ expressed in terms of $\phi$ using Fick's law, Eq. (10.42), this equation becomes the diffusion equation

$$-D \frac{d^2 \phi(x)}{dx^2} + \Sigma_a \phi(x) = \nu \Sigma_f \phi(x) + Q(x).$$  \hfill (10.47)

When the flux depends on three dimensions, Eq. (10.47) becomes

$$-D \nabla^2 \phi(r) + \Sigma_a \phi(r) = \nu \Sigma_f \phi(r) + Q(r)$$  \hfill (10.48)

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. This equation may be rearranged as

$$\nabla^2 \phi(r) + \frac{\nu \Sigma_f / \Sigma_a - 1}{D / \Sigma_a} \phi(r) = -\frac{Q(r)}{D}.$$  \hfill (10.49)

For the one-speed model, $\epsilon = 1$ and $p = 1$ since there are no fast neutron effects, and $k_{\infty} = \eta f = \nu \Sigma_f / \Sigma_a$. Then, with the diffusion length $L$ defined as $L^2 \equiv D / \Sigma_a$, Eq. (10.49) becomes

$$\nabla^2 \phi(r) + \frac{k_{\infty} - 1}{L^2} \phi(r) = -\frac{Q(r)}{D}.$$  \hfill (10.50)

There are two main classes of problems that can be addressed with the neutron diffusion equation. First, a nuclear engineer may want to know the spatial distribution of the flux density $\phi(r)$ produced by a specified external neutron source $Q(r)$. This is known as the fixed source problem. For example, a PuBe ($\alpha$,n) source may be stored in some medium, and the neutron field established in the medium by the source is needed (1) to estimate the rate of creation of radioactive material caused by neutron absorption and (2) to estimate the number of neutrons escaping from the storage container.
The second type of problem is the so-called criticality problem. Here the nuclear engineer seeks to determine the conditions under which a given mass of material is critical. In a critical configuration, a steady state neutron field can be maintained without the need for external source neutrons \((Q = 0)\), i.e., the chain reaction is sustained by fission neutrons alone. Thus the analyst seeks the conditions under which Eq. (10.48) has a non-zero steady-state solution when the external source \(Q\) is absent.

Examples of these two types of neutron diffusion problems are given below.

### 10.9.1 An Example Fixed-Source Problem

Consider an infinite homogeneous medium of non-fissionable material (i.e., \(\Sigma_f = 0\)) which contains an infinite plane source of neutrons with a strength such that each unit area of the source plane emits \(S_0\) neutrons isotropically in a unit time. Clearly, the neutron flux density \(\phi(x)\) depends on only the distance \(x\) from the source plane, which we place in the \(y-z\) plane at \(x = 0\). The diffusion equation for \(\phi(x)\) to the right of the source is

\[-D \frac{d^2\phi(x)}{dx^2} + \Sigma_a \phi(x) = 0, \quad x > 0 (10.51)\]

or, upon dividing through by \(-D\),

\[\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = 0, \quad x > 0. (10.52)\]

Because of the problem symmetry, the flux to the left of the source is the mirror image of the flux density to the right, i.e., \(\phi(x) = \phi(-x)\).

The most general solution of Eq. (10.52), a homogeneous, second-order, ordinary, differential equation, is

\[\phi(x) = Ae^{-x/L} + Ce^{x/L} (10.53)\]

where \(A\) and \(C\) are arbitrary constants whose values must be obtained by applying this general solution to boundary conditions for the problem. For example, as \(x \to \infty\), the flux density must vanish since all neutrons will be absorbed before they can diffuse an infinite distance from the source. The only way in which the right-hand side of Eq. (10.53) can vanish at infinity is for the constant \(C\) to vanish. Thus the form of the flux for this problem must have the form

\[\phi(x) = Ae^{-x/L}. (10.54)\]

To determine \(A\) we need a condition on \(\phi\) at the other boundary, i.e., next to the source plane. Consider, a unit area parallel to the source and just to the right of it. Through this unit area, many neutrons will pass in unit time, some traveling to the right and some to the left. However, because of symmetry, those neutrons diffusing from the right halfspace through the unit area into the left halfspace are exactly balanced by those neutrons diffusing in the other direction. Thus, the net flow \(J_x(0^+)\) into the right halfspace is due only to neutrons emitted by the source plane into the right halfspace, namely \(S_0/2\). With Fick's law, this condition is expressed mathematically as

\[J_x(0^+) = -D \left. \frac{d\phi(x)}{dx} \right|_{x=0^+} = \frac{S_0}{2}. (10.55)\]
Substitution of $\phi(x) = Ae^{-x/L}$ into this condition and solving the resulting equation for $A$ yields $A = (S_0 L)/(2D)$. With this value for $A$, using the problem symmetry, the flux density is

$$\phi(x) = \frac{S_0 L}{2D} e^{-|x|/L}. \tag{10.56}$$

From this explicit solution for the flux density, we note that the diffusion length $L$ determines the gradient at which the neutron flux density decreases with distance from the source.

### 10.9.2 An Example Criticality Problem

Consider a bare homogeneous slab of multiplying material ($\Sigma_f > 0$) which is infinite in the y- and z-directions and has a thickness $a$ in the x-direction. If there is no external source ($Q = 0$), the flux density is given by Eq. (10.50), namely, with the origin at the center of the slab,

$$\frac{d^2\phi(x)}{dx^2} + B^2_{\text{mat}} \phi(x) = 0 \quad -\frac{a}{2} < x < \frac{a}{2} \tag{10.57}$$

where the material buckling is $B^2_{\text{mat}} \equiv (\kappa_{\infty} - 1)/L^2$. For the slab to be critical, $\kappa_{\infty}$ must be greater than unity to allow for leakage, and hence $B^2_{\text{mat}} > 0$.

The solution of Eq. (10.57) also requires us to specify appropriate boundary conditions. Near the slab faces, the flux density can be expected to decrease very rapidly as $x \to \pm a/2$ because of the very strong probability that a neutron diffusing near the surface will leak from the slab, never to return. Hence, if the flux profile were extrapolated past the slab surface, it would appear to vanish at some extrapolation distance beyond the surfaces. It can be shown that this extrapolation distance is generally very small compared to the size of the reactor, and so, for this example, we assume that the flux density vanishes at the actual surfaces, i.e., we impose on Eq. (10.57) the boundary conditions $\phi(\pm a/2) = 0$.

Generally Eq. (10.57), subject to these boundary conditions, has only the trivial $\phi = 0$ solution. This makes sense since, for a slab of arbitrary composition, it is very unlikely we would just happen to choose the exact thickness to make the slab critical. Mathematically, ignoring the boundary conditions for the moment, the most general solution of Eq. (10.57) is

$$\phi(x) = A \cos(B_{\text{mat}} x) + C \sin(B_{\text{mat}} x) \tag{10.58}$$

where $A$ and $C$ are arbitrary constants. Because the critical flux profile must be symmetric about the origin (letting $x \to -x$ gives the same slab), the constant $C$ must be identically zero. Finally, we apply the boundary conditions to this general solution, namely

$$\phi(\pm a/2) = A \cos(\pm B_{\text{mat}} a/2) = 0. \tag{10.59}$$

From this requirement, we see that either $A = 0$ (which gives us the trivial $\phi = 0$ solution) or the argument $B_{\text{mat}} a/2$ must equal an odd multiple of $\pi/2$. Thus, to obtain a non-trivial solution, we require

$$B_{\text{mat}} = \frac{n\pi}{a} \quad n = 1, 3, 5, \ldots, \tag{10.60}$$
which yields the non-trivial solutions

$$\phi(x) = A \cos \left( \frac{n\pi}{a} x \right), \quad n = 1, 3, 5, \ldots \quad (10.61)$$

From this infinity of non-trivial solutions, we pick the $n = 1$ solution for two reasons. First, we want $B_{\text{mat}}^2$ (and hence $k_{\infty}$) to be as small as possible. Second, and more important, all the non-trivial solutions for $n > 1$ become negative in some regions between $-a/2$ and $a/2$, a physical impossibility since a negative neutron density is meaningless. The only physically realistic non-trivial solution, which is the critical flux profile $\phi_c$, is

$$\phi_c(x) = A \cos \left( \frac{\pi}{a} x \right), \quad (10.62)$$

and occurs whenever the following criticality condition is achieved:

$$B_{\text{mat}} = \left( \frac{n\pi}{a} \right). \quad (10.63)$$

Notice that the left-hand side of this criticality condition depends on only the material properties of the slab, while the right-hand side depends on only the geometry (a slab and its thickness). The right-hand side is often referred to as the geometric buckling $B_g$. Thus, the criticality condition, as seems reasonable, balances material and geometric properties of the core. For a bare core of any shape, the criticality condition can be shown to be

$$B_{\text{mat}} = B_g, \quad (10.64)$$

where $B_g$ depends only on the shape and size of the assembly. The critical flux profile and geometric bucklings for some simple bare homogeneous cores are listed in Table 10.6.

### 10.9.3 More Detailed Neutron-Field Descriptions

For detailed analyses of reactor cores it is necessary to account for the wide distribution of neutron energies. The one-speed diffusion equation, Eq. (10.47) or Eq. (10.48) can be easily generalized to give the energy-dependent flux density $\phi(r, E)$. This energy-dependent diffusion equation is then most often approximated by a series of coupled, one-speed, diffusion equations, known as the multigroup diffusion equations, by averaging the energy-dependent equation over a series of contiguous energy intervals (or groups) that cover the entire energy range of neutrons encountered in a reactor. Numerical methods must usually be used to solve these energy-multigroup diffusion equations.

On occasion, the diffusion equation is inadequate, for example, when a detailed flux profile near a control rod is needed. The difficulty with the diffusion equation is that Fick’s law is not exact, but rather, it is only an approximate description of how neutrons behave. In most instances, it is a good approximation, as verified by its wide-spread use for routine reactor analyses; however, for certain specialized calculations it is quite inaccurate.

For such difficult problems, it is necessary to use the neutron transport equation, an “exact” equation which gives the neutron flux density, $\phi(r, E, \Omega)$, as a function of position, energy and directions of travel $\Omega$ of the neutrons. The transport
Table 10.6. Critical flux profiles and bucklings for some homogeneous bare assemblies. Origin is at the assembly's center.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Dimensions</th>
<th>Flux Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slab</td>
<td>Thickness</td>
<td>$A \cos \left( \frac{\pi x}{a} \right) \left( \frac{\pi}{a} \right)^2$</td>
</tr>
<tr>
<td>Rect. Parallelepiped</td>
<td>$a \times b \times c$</td>
<td>$A \cos \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi y}{b} \right) \cos \left( \frac{\pi z}{c} \right) \left( \frac{\pi}{a} \right)^2 + \left( \frac{\pi}{b} \right)^2 + \left( \frac{\pi}{c} \right)^2$</td>
</tr>
<tr>
<td>Sphere</td>
<td>Radius $R$</td>
<td>$A \frac{1}{r} \sin \left( \frac{\pi r}{R} \right) \left( \frac{\pi}{R} \right)^2$</td>
</tr>
<tr>
<td>Infinite Cylinder</td>
<td>Radius $R$</td>
<td>$AJ_0 \left( \frac{2.405r}{R} \right) \left( \frac{2.405}{R} \right)^2$</td>
</tr>
<tr>
<td>Cylinder</td>
<td>Rad. $R$, Ht $H$</td>
<td>$AJ_0 \left( \frac{2.405r}{R} \right) \cos \left( \frac{\pi z}{H} \right) \left( \frac{2.405}{R} \right)^2 + \left( \frac{\pi}{H} \right)^2$</td>
</tr>
</tbody>
</table>

The equation is very difficult to solve for practical problems, and, consequently, computationally expensive methods, such as the discrete-ordinates method, must be used. An alternative approach to solving neutron transport problems is to use numerical simulation of the random interactions experienced by the neutrons. This alternative approach, called the Monte Carlo method, is also very computationally expensive for practical reactor analyses.

10.10 Addendum 2: Kinetic Model with Delayed Neutrons

To account properly for delayed neutrons, it is necessary to account for the time variation of the delayed neutron precursors as well as the number of neutrons $n(t)$ in the core. For simplicity, we assume a one delayed-neutron group approximation, i.e., all the delayed neutron precursors have the same half-life or decay constant $\lambda$.

Let $C(t)$ be the total number of delayed neutron precursors in the core at time $t$.

In each neutron cycle, $k_{eff} n(t)$ new fission neutrons are eventually produced, $(1 - \beta)k_{eff} n(t)$ as prompt neutrons at the end of the cycle and $\beta k_{eff} n(t)$ as delayed neutrons which appear upon the decay of the precursors that are produced during the cycle. Thus, in each cycle, $\beta k_{eff} n(t)$ precursors are created, and, since each cycle takes $\ell'$ seconds to complete, the number of precursors created per unit time is $\beta k_{eff} n(t)/\ell'$. The rate of disappearance of precursors by radioactive decay is $\lambda C(t)$. The net rate of increase in the number of precursors is the production rate minus the decay rate, i.e.

$$\frac{dC(t)}{dt} = \frac{\beta k_{eff}}{\ell'} n(t) - \lambda C(t).$$

(10.65)

In each cycle $(1 - \beta)k_{eff} n(t)$ prompt neutrons are produced and $n(t)$ neutrons disappear. Thus, the net rate of increase by prompt neutrons is $k_{eff}(1 - \beta) - \lambda C(t)$.
1\frac{n(t)}{\ell'}$. In addition, neutrons are produced by the decay of the delayed neutron precursors at the rate $\lambda C(t)$ and by non-fission sources at a rate $S(t)$. The rate of increase in the neutron population is thus the sum of these three neutron production mechanisms, i.e.,

$$\frac{dn(t)}{dt} = \frac{1}{\ell'} k_{\text{eff}} (1 - \beta) - 1)n(t) + \lambda C(t) + S(t)$$

$$= k_{\text{eff}} \left[ k_{\text{eff}} - 1 \right] n(t) + \lambda C(t) + S(t). \tag{10.66}$$

Finally, we define the effective neutron lifetime $\ell = \ell' / k_{\text{eff}}$ and the reactor reactivity $\rho$ as

$$\rho = \frac{k_{\text{eff}} - 1}{k_{\text{eff}}}. \tag{10.67}$$

The reactivity, like $k_{\text{eff}}$, is a measure of the critical state of the reactor. A positive reactivity indicates a supercritical system, and a negative value a subcritical system, while a critical system has zero reactivity. With these definitions, the kinetic equations Eqs. (10.65) and (10.66) become

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\ell} n(t) + \lambda C(t) + S(t) \tag{10.68}$$

and

$$\frac{dC(t)}{dt} = \frac{\beta}{\ell} n(t) - \lambda C(t). \tag{10.69}$$

These coupled kinetic equations are readily generalized to the case of $G$ groups of delayed neutron precursors. Each precursor group has its own kinetic equation, and the delayed neutron production term in the neutron equation is the sum of the decay rates of all the precursors. The result is

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\ell} n(t) + \sum_{i=1}^{G} \lambda_i C_i(t) + S(t) \tag{10.70}$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\ell} n(t) - \lambda_i C_i(t), \quad i = 1, \ldots, G$$

These kinetic equations are known as the point reactor kinetic equations and are widely used to describe the transient response of reactors.

For low-power research reactors, which maintain nearly constant composition over a long time, it is customary in the U.S. to measure reactivity in units of “dollars” defined as $k(\$) \equiv \rho / \beta$. Multiplication of Eqs. (10.70) by $(\ell / \beta)$ and defining

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\( \bar{C}_i(t) \equiv (\ell/\beta)C_i(t) \) yields the equivalent form of the point reactor kinetic equations

\[
\frac{\ell}{\beta} \frac{dn(t)}{dt} = [k - 1]n(t) + \sum_{i=1}^{G} \lambda_i \bar{C}_i(t) + \frac{\ell}{\beta}S(t) \\
\frac{d\bar{C}_i(t)}{dt} = a_i n(t) - \lambda_i \bar{C}_i(t), \quad i = 1, \ldots, G
\]  

(10.71)

where the relative precursor yield \( a_i = \beta_i/\beta \). Noted that \( \sum_{i=1}^{G} a_i = 1 \).

### 10.11 Addendum 3: Solution for a Step Reactivity Insertion

Consider a source-free reactor operating at a steady power level \( n_0 \). At \( t = 0 \) the reactivity is changed from its steady-state value of zero to \( k_o \), i.e.,

\[
k(t) = \begin{cases} 
0 & t < 0 \\
k_o & t \geq 0
\end{cases}
\]  

(10.72)

For \( t > 0 \) and a one delayed-neutron group model (\( a_i = 1 \)), Eqs. (10.71) become

\[
\frac{\ell}{\beta} \frac{dn(t)}{dt} = [k_o - 1]n(t) + \lambda \bar{C}(t) \\
\frac{d\bar{C}(t)}{dt} = n(t) - \lambda \bar{C}(t).
\]  

(10.73)

(10.74)

Solutions to these two first-order, coupled, ordinary differential equations, are of the form \( n(t) = Ae^{\omega t} \) and \( \bar{C}(t) = Be^{\omega t} \). Substitution of these assumed forms into Eq. (10.74) yields

\[
B = \frac{A}{(\lambda + \omega)}.
\]  

(10.75)

Substitution of this result into Eq. (10.73) gives

\[
\frac{\ell}{\beta} \omega A e^{\omega t} = (k_o - 1)Ae^{\omega t} + \frac{\lambda A}{(\lambda + \omega)}e^{\omega t}
\]  

(10.76)

or, upon simplification,

\[
A \left\{ \frac{\ell}{\beta} \omega + \frac{\omega}{\lambda + \omega} - k_o \right\} = 0.
\]  

(10.77)

Thus for \( A \neq 0 \), we see that \( \omega \) must be such that

\[
\omega \left[ \frac{\ell}{\beta} + \frac{1}{\lambda + \omega} \right] = k_o
\]  

(10.78)

This equation, which determines the permissible values of \( \omega \) is known as the "in-hour" equation, since \( \omega \) has units of inverse time, typically measured in "inverse hours."
For the one delayed-neutron-group model, the inhour equation is a quadratic equation in \( \omega \), i.e., there are two permissible values of \( \omega \). Simplification of Eq. (10.78) gives

\[
\omega^2 + \omega \left[ \lambda + \frac{\beta}{\ell} (1 - k_\alpha) \right] - k_\alpha \frac{\beta}{\ell} = 0. \tag{10.79}
\]

The two solutions \( \omega_1 \) and \( \omega_2 \) of this quadratic equation are

\[
\omega_{1,2} = \frac{1}{2} \left\{ - \left( \lambda + \frac{\beta}{\ell} (1 - k_\alpha) \right) \pm \sqrt{\left( \lambda + \frac{\beta}{\ell} (1 - k_\alpha) \right)^2 + 4 k_\alpha \frac{\beta}{\ell}} \right\}. \tag{10.80}
\]

Thus, the most general solution of Eqs. (10.73) and (10.74) is the linear combination of the two possible solutions, namely,

\[
\begin{align*}
n(t) &= A_1 e^{\omega_1 t} + A_2 e^{\omega_2 t} \tag{10.81} \\
\tilde{C}(t) &= \frac{A_1}{\lambda + \omega_1} e^{\omega_1 t} + \frac{A_2}{\lambda + \omega_2} e^{\omega_2 t}. \tag{10.82}
\end{align*}
\]

To determine the arbitrary constants \( A_1 \) and \( A_2 \), we use the initial conditions

\[
n(0) = n_0 \tag{10.83}
\]

and, from Eq. (10.73) and Eq. (10.74),

\[
\frac{\ell}{\beta} \left. \frac{dn(t)}{dt} \right|_{t=0} = (k_\alpha - 1)n_0 + k_\alpha n_0 = k_\alpha n_0. \tag{10.84}
\]

Substitution of Eqs. (10.81) and (10.82) into these two initial conditions gives the following two algebraic equations for \( A_1 \) and \( A_2 \):

\[
\begin{align*}
A_1 + A_2 &= n_0 \tag{10.85} \\
\omega_1 A_1 + \omega_2 A_2 &= \frac{\beta}{\ell} k_\alpha n_0. \tag{10.86}
\end{align*}
\]

The solution of these equations is

\[
\begin{align*}
A_1 &= \frac{\beta k_\alpha / \ell - \omega_2}{\omega_1 - \omega_2} n_0 \tag{10.87} \\
A_2 &= \frac{\omega_1 - \beta k_\alpha / \ell}{\omega_1 - \omega_2} n_0. \tag{10.88}
\end{align*}
\]

Substitution of \( A_1 \) and \( A_2 \) into Eq. (10.81) gives

\[
n(t) = n_0 \left\{ \frac{\beta k_\alpha / \ell - \omega_2}{\omega_1 - \omega_2} e^{\omega_1 t} + \frac{\omega_1 - \beta k_\alpha / \ell}{\omega_1 - \omega_2} e^{\omega_2 t} \right\}. \tag{10.89}
\]

For the special case of “small” reactivities, i.e., \( (1 - k_\alpha)^2 >> 4(\ell/\beta)k_\alpha \lambda \) the following approximations hold:

\[
\begin{align*}
\omega_1 &\approx \frac{\lambda k_\alpha}{1 - k_\alpha} \quad \text{and} \quad \omega_2 \approx -\frac{1 - k_\alpha}{\ell/\beta} < 0 \tag{10.90} \\
A_1 &\approx \frac{n_0}{1 - k_\alpha} \quad \text{and} \quad A_2 \approx -\frac{n_0 k_\alpha}{1 - k_\alpha}. \tag{10.91}
\end{align*}
\]
With this approximation, the transient variation of the neutron population is

$$n(t) \simeq \frac{n_0}{1 - k_0} \left\{ \exp \left[ \frac{\lambda k_0 t}{1 - k_0} \right] - k_0 \exp \left[ -\frac{(1 - k_0)t}{\ell/\beta} \right] \right\}. \quad (10.92)$$

**BIBLIOGRAPHY**


**PROBLEMS**

1. In a liquid metal fast breeder reactor, no neutron moderation is desired and sodium is used as a coolant to minimize fission-neutron thermalization. How many scatters with sodium, on the average, would it take for 2-Mev neutrons to reach an average thermal energy of 0.025 eV? HINT: review Section 6.5.1.

2. Discuss the relative merits of water and graphite for use in a thermal reactor.

3. List five desirable properties of a moderator for a thermal reactor. Explain the importance of each property.

4. What is the thermal fission factor $\eta$ for 5 atom-% enriched uranium?

5. What atom-% enrichment of uranium is needed to produce a thermal fission factor of $\eta = 1.85$?

6. Plot the thermal fission factor for uranium as a function of its atom-% enrichment in $^{235}$U.

7. A soluble salt of fully enriched uranium is dissolved in water to make a solution containing $1.5 \times 10^{-3}$ atoms of $^{235}$U per molecule of water.

   (a) Explain why $\epsilon p \simeq 1$ for this solution.

   (b) What is $k_{\infty}$ for this solution? Neglect any neutron absorption by other elements in the uranium salt.
(c) This solution is used to fill a bare spherical tank of radius $R$. Plot $k_{\text{eff}}$ versus $R$ and determine the radius of the tank needed to produce a critical reactor.

8. Consider a homogeneous, bare, spherical, source-free, critical, uranium-fueled reactor operating at a power $P_0$. Explain how and why the power increases, decreases, or remains unchanged as a result of each of the separate changes to the reactor.

(a) The reactor is deformed into the shape of a football (ellipsoid).
(b) A person stands next to the core.
(c) The temperature of the core is raised.
(d) A neutron source is brought close to the core.
(e) An energetic electron beam impacts the core.
(f) The reactor is run at high power for a long time.
(g) The core is launched into outer space.
(h) A sheet of cadmium is wrapped around the core.
(i) The enrichment of the fuel is increased.

9. For a given amount of multiplying material with $k_\infty > 1$, what is the shape of a bare core with the smallest mass of this material? Explain.

10. If the uranium fuel enrichment in a reactor is increased, what is the effect on $k_{\infty}$? Explain.

11. Consider a homogeneous mixture of fully enriched $^{235}$U and graphite. Plot $k_\infty$ versus $N^{235}/N^C$. What is the fuel-to-moderator ratio that yields the maximum value of $k_\infty$?

12. What is the optimum fuel-to-moderator ratio of a homogeneous mixture of $^{235}$U and (a) light water, (b) heavy water, (c) beryllium, and (d) graphite to produce a mixture with the maximum $k_\infty$? Data: the thermal (0.0253 eV) absorption cross sections for water, heavy water, beryllium and carbon are 0.664, 0.00133, 0.0092 and 0.0034 b per molecule or atom, respectively.

13. A spherical tank with a radius of 40 cm is filled with a homogeneous mixture of $^{235}$U and light water. The mixture has a moderator-to-fuel ratio $N^{H_2O}/N^{235}$ of 800. (a) What is $k_\infty$ of the mixture? (b) What is $k_{\text{eff}}$ for this bare core?

14. What should the radius of the tank in the previous problem be to produce a critical configuration? What is the critical mass of $^{235}$U needed?

15. A control rod is dropped into a critical, source-free, uranium-fueled reactor and the asymptotic period of the resulting exponentially decreasing power is observed to be -2 s. (a) What is the value of $k_{\text{eff}}$ of the reactor after the control rod drop? (b) What was the reactivity insertion in dollars?
16. What is the asymptotic period resulting from a reactivity insertion of (a) 0.08$ and (b) -0.08$?

17. Following a reactor scram, in which all the control rods are inserted into a power reactor, how long is it before the reactor power decreases to 0.0001 of the steady-state power prior to shutdown?

18. At time $t = 0$ a reactivity of 0.15$ is inserted into a critical reactor operating at 100 W. How long is it before the reactor power reaches 1 MW?

19. A reactivity insertion into an initially critical reactor operating at steady state causes the power to increase from 100 W to 10 kW in 6 minutes. What was the value of the reactivity insertion in $?

20. Explain how a decrease in the boiling rate inside a boiling water reactor affects the reactivity of the reactor.

21. Explain why it is reasonable that $^{135}$Xe should have a very large cross section for neutron absorption.

22. Following a reactor shutdown ($\phi_0 \rightarrow 0$), show from Eq. (10.33) that the time $t_m$ for $^{135}$Xe to reach a maximum is given by

$$t_m = -\frac{1}{\lambda_i - \lambda_x} \ln \left\{ r \left[ 1 + (1 - r) \frac{X(0)}{I(0)} \right] \right\},$$

where $X(0)$ and $I(0)$ are arbitrary $^{135}$Xe and $^{135}$I concentrations at shutdown and $r = \lambda_x / \lambda_i$. From this result show, that for an increase in $^{135}$Xe following shutdown, the initial values must satisfy

$$\frac{X(0)}{I(0)} \leq \frac{\lambda_i}{\lambda_x} = 1.38.$$

23. In a fast reactor, does $^{135}$Xe produce feedback? Why?

24. Many phenomena produce reactivity feedback. Within orders of magnitude, over what time interval would you expect the following reactivity feedback causes to take effect? (a) temperature increase in the fuel, (b) temperature increase in the moderator/coolant, (c) $^{135}$Xe increase, (d) fuel burnup, (e) boiling in the core, and (f) increased coolant flow through the core.